Using Math Literacy Strategies in Calculus: Amplifying Undergraduates' Understanding of Partial Derivatives

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Mathematical literacy is a critical component of teaching mathematics as its own language. In this paper, we share the use of a writing to learn mathematics (WTLM) activity to help college students communicate their mathematical thinking and understanding. We conducted an embedded single-case study with two sections of undergraduate students enrolled in a Calculus II course focused on integral and multivariable calculus topics. Three WTLM activities were integrated into the course to provide opportunities for students to explain their symbolic, procedural, and conceptual understandings of partial derivatives and related topics. In this paper we share the framework and tools developed for this study and the data analysis plan and preliminary analysis steps.

Keywords: partial derivatives, concept definition, concept image, math literacy

Traditional teaching practices often limit how students are asked to communicate their mathematical understanding. When considering mathematics as a language, and teaching it as such, we can use practices like those used in language learning to develop mathematical communication skills by paying attention to notations as our alphabet, concepts as our ideas, and mathematical statements as sentences (Alfaro Viquez and Joutsenlahti, 2020; Bruun et al., 2015; Roepke and Gallagher, 2015). Considering mathematics as a language allows instructors to focus on developing students' ability to communicate mathematically by "unpacking" and "repacking" mathematical concepts, procedures, and representations (Viquez & Joutsenlahti, 2020). This ability is one of the components of mathematical literacy, which is "an individual's capacity to formulate, employ, and interpret mathematics in various contexts[including] reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena" (OECD, 2018, p. 67).

In this study, we examine the use of writing to learn mathematics (WLTM) tools to provide opportunities for students to share their concepts beyond traditional representations and procedures and build their mathematical literacy. We focus specifically on the concept of partial derivatives as introduced and used in a combination Calculus II and III course. This topic is important to address in the mathematics classroom because of how applicable partial derivatives are in both math and other STEM disciplines (Becker and Towns, 2012; Bucy et al., 2007; Emigh and Manogue, 2017). We addressed the research questions: (1) What are students' concept definitions and concept images when initially learning about partial derivatives in calculus? and (2) How do topics that use partial derivatives add to students' conceptual understandings? This work is part of Cuadra's larger study using three distinct WTLM activities as data collection sources. In this paper, we share the framework and the data collection tools that we have designed for this study as well as the analysis plan.

Framework

One way of strengthening students' understanding of mathematical phenomena is by associating a concept definition or properties with a strong concept image. A *concept definition* is "a verbal definition that accurately explains the concept in a non-circular way" (Vinner, 1983,

p. 293). A concept definition is similar to the usual notion of a definition in mathematics, except the concept definition includes attributes and properties that the writer of that definition deems important. A *concept image* “a set of properties together with a mental picture” and generally refers to a visual representation or depiction of a particular concept (Vinner, 1983, p. 293). This image may be an illustration, diagram, graphic, or any visual medium that conveys and communicates the essence of a specific idea or abstract notion. Focusing on concept definitions and concept images, we can restructure the teaching of mathematics to become conceptual as well as procedural. As Tall (1992) states ``the individual's method of thinking about mathematical concepts depends on more than just the form of words used in a definition'' (p. 496). Hence, students’ experiences factor in when they are developing ideas about a mathematical concept. By attending to concept definitions and concept images, students can develop a deeper and more meaningful connection with the material.

Zandieh (2000) created a framework to study students’ concept definitions and concept images of derivatives in a first-year calculus course. Her framework combined different representations of derivatives (e.g., graphical, verbal, symbolic, and physical) and process-object layers of the derivative. The latter come from Sfard’s (1992) idea of processes as operations on previously established objects. Zandieh included process-object layers of the concept of derivatives (e.g., ratio, limit, and function) that can be viewed both as dynamic operational processes and as static structures objects.

Emigh and Manogue (2017) used Zandieh’s framework to analyze students’ interpretations of partial derivatives in undergraduate math and physics courses through interviews. Two common interpretations from the students were that partial derivatives relate to slope and change. Roundy et al. (2015) also modified both the content and the structure of Zandieh’s framework to study students’ concept images of partial derivatives and included content-specific items like experimentally measured quantities. Their framework was developed using what they learned about experts’ concept images of partial derivatives.

Following the work done by Roundy et al. (2015), we adapted Zandieh’s original framework by using the representations and process-object layers related to partial derivatives as students first learn about them in calculus. To determine the different process-object layers, it is helpful to think of the underlying mathematical concepts that support an understanding of partial derivatives. As partial derivatives are an extension of single-variable derivatives, the original process-object layers for single-variable derivatives found in Zandieh’s (2000) framework are still applicable for partial derivatives, namely (a) ratio, (b) limit, and (c) function. In addition, we consider (d) single-variable derivatives as an important added process-object layers of partial derivatives. One key modification is the incorporation of covariation reasoning into the existing framework. as this is a critical underlying concept that supports students’ understanding of partial derivatives. Mkhatshwa (2021) used a covariational reasoning framework when studying students’ procedural understanding of quantities represented by partial derivatives and found that students tended to shift from interpreting them as rates to interpreting them as amounts. The framework in Figure was used to guide the research design and analysis for this study.

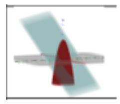
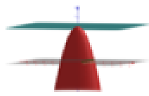
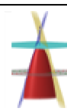
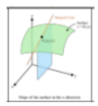
Process – object layer	Graphical	Verbal	Symbolic / Numerical	Physical
	Slope	***Rate of Change***	***Independence of Variables / Ratios of Change***	***Optimization***
Ratio		**Average rate of change**	**$\Delta f/\Delta x$ numerically**	
Limit		**Instantaneous rate of change**	**$\Delta f/\Delta x$ with Δx small**	
Function		**Rate of change at any point/time**	**$\partial f/\partial x$ depending on x**	**Extrema analytically**
Derivative		**Rate of change of f as x changes**	**Treat y as a constant when applying the differentiation rules**	**Extrema graphically**
Covariation	**Direction**	**Coordination**	**Quantitative Coordination**	
Non-layer	**“slope”**	**“rate of change”**	**“independence of variables”**	**“optimization”**

Figure 1. Framework guiding the research design for this study

Methods

Participants for this study included 43 undergraduate students enrolled in an applied calculus course that combined topics from both Calculus II and Calculus III curricula. Cuadra served as the teacher-researcher for two sections of this course.

This study is designed as an embedded single-case study to examine students’ understanding of partial derivatives (Yin, 2003). The single case of focus is calculus students’ understanding of the concept of partial derivatives. This study is bounded by the topics covered within the Applied Calculus II course and the students participating in the course. Since mathematics is a cumulative subject, it is important to look at the building blocks that encompass partial derivatives, such as single-variable derivatives and functions of several variables, as embedded cases that provide richer information on students’ understanding of partial derivatives. In addition, topics that are dependent on deep understanding of partial derivatives including higher-order partial derivatives, extrema(s) of two variable functions, Lagrange multipliers, and double integrals also serve as avenues of inquiry as these applications of the concept can influence understanding. Students used the data collection tools below for each of these building blocks and application topics as well as for the specific concept of partial derivatives. We have a complete set of data from 43 students collected over the course of a semester.

Data Collection

Data was collected using three WTLM math literacy activities intended to help students communicate their mathematical thinking in math courses (Porter & Masingila, 2000). WTLM gives students an opportunity to reflect on their own thinking about the mathematical ideas covered in the class and to communicate their understandings. For this study, Cuadra developed three tools: (a) “The Important” Prompt, (b) an Aspects of Mathematical Phenomenon chart, and (c) learning logs. Roepke and Gallagher (2015) adopted the “The Important” prompt from

Margaret Wise Brown's *The Important Book* (1949) for usage in mathematics classes. In Brown's book, she takes everyday objects and describes what makes each object special. Roepke and Gallagher (2015) adopted this idea in the following activity. First, the students complete the sentence "The important thing about (insert topic) is..." by writing at least five things they found important about that mathematical topic and then write what they found "most important" about the topic. This activity allows students to think beyond the typical "compute" exercises and analyze an idea beyond computation.

The second WTLM tool is one we are calling the *Aspects of Mathematical Phenomenon (AMP)* Chart. It is based on the vocabulary development tool known as the Frayer model (Frayer et al., 1969) which had four quadrants surrounding a featured vocabulary word written in the middle. In mathematics classrooms, scholars have adapted the Frayer model quadrant labels to include things like definitions (in your own words), facts, pictures, examples, and nonexamples (Kenney et al., 2005; Roepke & Gallagher, 2015). Since students are not accustomed to being asked to share a concept image, we have extended this kind of model by creating the AMP Chart to allow students to unpack the aspects of a mathematics concept across six cells and captures a snapshot of their concept definition and concept image. Figure 2 shows an example of a completed AMP chart.

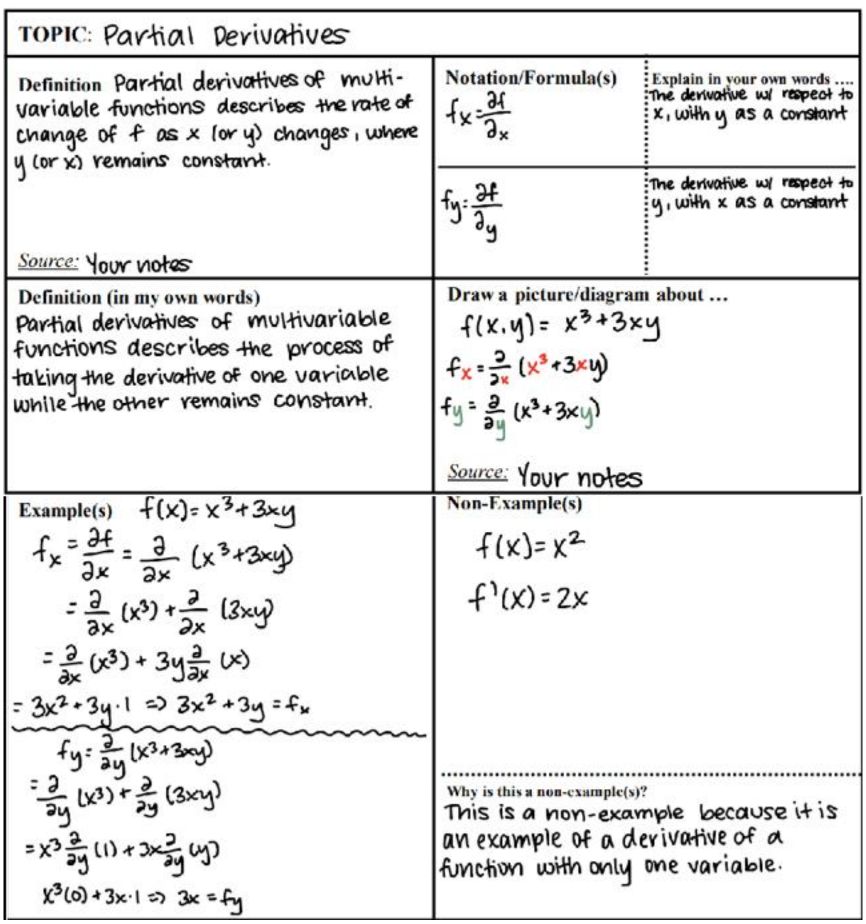


Figure 2. Example of a completed AMP chart by a Calculus II student.

The final WLTM tool used were learning logs, or journals that serve as a collection of student-generated ideas that examine the thinking processes and conceptual understanding of the students (Carter et al., 1993, p. 87). A key reason to use learning logs that they create connections between language and mathematical literacy (Carter et al., 1993). Learning logs here are reflection prompts that students write to share how their understanding of partial derivative changes after they engage with different representations and applications of partial derivatives.

Data Analysis Plan

As this is a preliminary report, data analysis of the significant amount of data collected is ongoing. We intend to analyze the WTLM activities that students did for partial derivatives and then for topics dependent on partial derivatives (i.e., Lagrange multipliers, higher-order partials, extrema, and double integrals).

We are using a *concept mapping* approach for analysis (Butler-Kisber & Poldma, 2010; Daley, 2004; Moreno, Guthrie, & Strickland, 2023). For the AMP charts, for example, we cut up the chart into six individual cells and are making a concept map for each cell, looking for connections and distinctions across the students' responses. This approach to analysis allows us to consolidate the data into manageable pieces (Daley, 2004) and map our interpretations of the data as these emerge (Butler-Kisber & Poldma, 2010). We deconstruct the students' responses to see what connections the whole group (as our single case) is making to the mathematics topics. Looking across the resulting concept maps for each of the cells of the AMP charts then helps us build a picture of the concept images developing surrounding partial derivatives. A similar process will use the Important Prompt to build a representation of the concept definition of partial derivatives that emerges when students first learn about it. Finally, the learning logs allow us to look beyond a single snapshot of this concept definition and image and see how they are evolving as new topics related to partial derivatives are presented in the course. Figure 3 shows an example of one initial concept maps that we have generated after examining the "Definition" cell of the AMP Chart. Using the Framework described above in Figure 1 allows us to see which process-object layers the students are attending to in these cells.

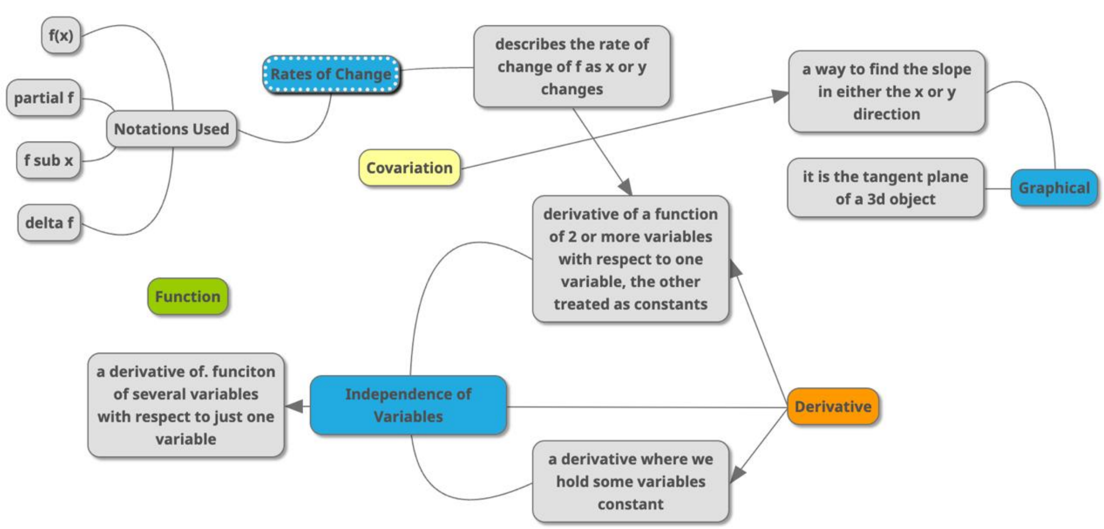


Figure 3. Concept map of students' definitions

Though data has been collected, we are currently just beginning analysis of the data and are eager to discuss ideas about this interesting work with the RUME community.

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