Elaboration on Lists and Listing Processes in Combinatorics

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Lists are a foundational way to reason about and solve counting problems, yet it is unclear to what extent their role should continue for advanced learners and what that role should be. I draw on case study data of two undergraduate students who used lists repeatedly as they worked through a sequence of combinatorial tasks. I demonstrate that the students' use of lists refined over time, and their representations developed from outcome-oriented to process-oriented, thus allowing them to draw on their knowledge of lists for increasingly sophisticated problems. Further, the forms of their solutions were impacted by their use of lists. I argue that these data shed light on broader questions of the role of representations in solutions to combinatorics problems as students develop combinatorial facility.

Keywords: combinatorics, combinatorial reasoning, lists of outcomes, semiotics

Introduction and Relevant Literature

Combinatorics is the mathematics of counting and enumeration. Finding the solution to counting problems often involves reasoning about and quantifying structure in the set of outcomes being counted (Lockwood, 2014). At the same time, there are few institutional means to represent those sets of outcomes in ways that are conducive to connecting structure to expressions of cardinality (Wasserman, 2019), despite the demonstrated positive impact of using representations as tools for combinatorial reasoning (Maher & Yankelewitz, 2010; Maher et al., 2010). Listing outcomes and reasoning about how to list the outcomes has been shown to enrich student combinatorial facility across grade level (De Chenne & Lockwood, 2022; English, 1993; Halani, 2012; Lockwood & De Chenne, 2020, 2021; Lockwood & Gibson, 2016; Montenegro et al., 2021; Tillema, 2018). One drawback of listing is that it can be intractable for larger problems. Lockwood and De Chenne (2020, 2021) address this issue by having students write computer programs to list the outcomes, and then reason about their algorithm and the produced list of outcomes to find a solution to the counting problem. However, De Chenne and Lockwood (2022) showed that even when students articulated a process to list the set of outcomes in a computer program, they were sometimes unable to articulate the same structure in the produced list of outcomes. These discrepancies in student reasoning call into question the comparative roles of lists themselves (inscribed representations) and reasoning about how to list when finding solutions to counting problems.

This paper elaborates on lists and listing processes in combinatorics, and their connections to how students find solutions and what form these solutions take. I draw on case study data of a pair of undergraduate students as they worked on a sequence of counting problems intended to elicit the use of inscribed representations. I focus on these two students because they regularly engaged in listing processes across six interviews, and importantly their use of lists was often challenged by balancing tractability with providing key combinatorial insight. Thus, these data demonstrate affordances of lists and listing process while also providing insight into how they can be applied in increasingly challenging and sophisticated scenarios. I seek to answer the following research question: *what roles do lists and listing processes have as students find solutions to combinatorics problems of increasing complexity?*

Theoretical Framework

I draw from Lockwood's (2013) model of combinatorial reasoning to describe broadly how students reason about and find solutions to counting problems. I then provide a theoretical characterization of lists and listing processes in combinatorics.

Model for combinatorial reasoning, and solutions to counting problems

Lockwood's (2013) model frames combinatorial reasoning in terms of sets of outcomes, formulas and expressions, and counting processes. Sets of outcomes are the collections of objects described in counting problems, the cardinalities of which are typically the solutions. Formulas and expressions are the mathematical expressions that yield some numerical value, most frequently used to express the cardinality of the set of outcomes. Counting processes are the real or imagined processes one engages with to enumerate, or count, the outcomes. While creating this model, Lockwood observed that some students approach counting as an activity of identifying the correct problem type (e.g., arrangement with repetition or selection without repetition) for the outcomes at hand, and applying a corresponding formula/expression. When one student was unsure which problem type was correct, she would "go off [her] gut" (Lockwood, 2014). Such an approach to counting poses a challenge for continued development as it minimizes understanding of why the formula applies.

Lockwood (2014) described a set-oriented perspective on counting as one where reasoning about the set of outcomes is an intrinsic component of solving counting problems. Entailed in this perspective is that sets of outcomes can be considered to contain underlying structure, and by leveraging and quantifying this structure the counter can identify a corresponding mathematical expression that provides the cardinality of the set of outcomes. Such a solution approach may employ a combinatorial formula such as $\frac{n!}{k!(n-k)!}$, but with the understanding of how and why that formula counts the set of outcomes. Wasserman (2019) reinforces this perspective, and he additionally argues that counting experts also have a complementary perspective where a mathematical expression represents a specific counting process. That is, the form of the solution represents what outcomes were counted and how they were counted. Drawing on this perspective, I characterize finding solutions to counting problems as determining a mathematical expression with two roles: yielding an integer value of the cardinality of the set of outcomes, and representing the structure of the outcomes used to determine this value. Hence, when examining the role of lists and listing processes in student solutions I will be examining the form of the solution as a way of understanding how students leveraged structure to find a mathematical expression of cardinality.

Lists and listing processes

Drawing from De Chenne and Lockwood (2022), I define a *listing process* to be a counting process in which one imagines how a set of outcomes can be enumerated, producing an ordered sequence. A *list* of outcomes is the artifact produced by enacting a listing process. Listing processes need not be able to produce a complete and totally ordered sequence of the outcomes, and I use the term *resolution* to describe the extent of the order in the resulting list. A listing process that demonstrates complete resolution is one where every outcome is created in a pre-defined order, whereas low resolution articulates a broad structure in a set but not substructures. For example, a low-resolution listing process might state how a set can be partitioned, but not how the outcomes in each part are ordered. I make no attempt to quantify resolution here, and I only use resolution as a means of comparing differences across the course of multiple interviews.

Methods

Participants and Data Collection

The data in this paper come from a larger study conducted at a large R1 university in the Northwest United States that investigated how undergraduate students use representations in their solutions to combinatorics problems. Hallie and Leah (pseudonyms) were two participants who worked as pair on a sequence of combinatorial tasks. The students completed six task-based interviews (9 hours total) over the course of one semester, and the data reported in this paper span the first five interviews. I chose to pair Hallie and Leah based on their similar educational background and their responses to the combinatorics problems presented in selection interviews. Both students were biological data science majors (Hallie was a senior and Leah was a sophomore), and neither student successfully used combinatorial formulas to solve the problems. Hallie had previously taken a discrete mathematics course with a section on combinatorics, but her responses in the selection interview indicated that she did not use formulas from that course to answer combinatorial tasks. Leah was currently enrolled a discrete mathematics course, but at the time of the interviews she had not completed the section on combinatorics.

I chose to present these data as a case study because Hallie and Leah repeatedly used lists and listing processes as foundational aspects of their solution. That is, lists and listing processes were fundamental to how they found solutions and were not used merely to communicate a solution found through other means. Additionally, over the course of the interviews Hallie and Leah adapted their lists and listing processes to increasingly complex problems with growing cardinalities. In doing so, they leveraged lists and listing processes with other structural information such as symmetry and recursion, which provides insight into how lists might be used as one tool used in conjunction with others. These data thus provide insight into how lists might be used specifically, and how external representations might be used more generally in combinatorics.

Tasks and Data Analysis

The task sequence presented to students in the interviews consisted of combinatorics problems that could be solved using arrangements with and without repetition, and selection without repetition. These problem types are consistent with other combinatorics education literature (see Lockwood & De Chenne (2020) for a broader discussion of problem types). However, I selected tasks to elicit the use of representations, and thus while the problems could be framed as arrangements and selections the literal outcomes described did not always match this framing. For example, the outcomes in the Three Cards Problem (described in the Results section) can be framed as a selection without repetition problem even if the literal outcomes described are ordered arrangements. Parts of tasks also explicitly asked students to describe how they would represent the outcomes.

The interviews were conducted in a private office and video and audio recorded. I provided students with writing materials, and their work was collected and scanned after each interview. From these scans I created a catalogue of representations, and in conjunction with the recordings of the interviews I used thematic analysis (Braun & Clarke, 2022) to identify what types of representations students used, categories of representations, and the roles of the representations in solutions. I focus here on Hallie's and Leah's use of lists and listing processes; however these two students used other representations across the six interviews.

Results

I describe three themes that emerged in Hallie and Leah's use of lists and listing processes. First is the balance between the resolution of a counting process and the affordance of a list for providing a means of reflecting on a counting process. The second the application of lists to solve part but not all of a problem by allowing for flexibility in what is being listed. The third is moving from representations of outcomes in lists to representations of listing processes more directly, and how those listing processes are reflected in the solution.

Resolution and Reflection

Hallie's and Leah's work on early tasks demonstrates how low-resolution listing processes could be used because the act of listing allowed them to reflect on and create lists in the moment. The first task Hallie and Leah completed was the Coin Flip Problem, which asked how they would record flipping a coin three times, how many ways there are to do so, and how their answers to these problems would change if the number of coins increased. For three coins, both Hallie and Leah created complete lists of the outcomes, although their listing strategies differed from each other. These lists are shown in Figure 1. Hallie's listing process entailed using two columns, where the second column was the mirror image of the first. This approach leveraged the symmetry of the outcomes; however, she did not have a pre-determined strategy for which outcomes would occur in which column. Rather, her approach was to write an outcome in the first column, write the mirror outcome in the second column, and then in the first column write another outcome that had not already appeared in the first or second column. Hallie concluded listing when she could not find an outcome that had not appeared in either column. She described this approach as "I kind of start where I separate at both extremes [HHH and TTT], and then get kinda to the middle."

Leah's approach was to structure the outcomes by the number of heads that occurred; that is, she listed all outcomes where three heads occurred, then all outcomes where two heads occurred, etc. While this provides a broad structure to the list of outcomes, the occurrence of each individual outcome within that structure was not predetermined, and she concluded each part of the list when she could not think of any unlisted outcomes. Hence, Hallie's and Leah's listing processes demonstrate a low-resolution strategy. Despite this, neither Hallie nor Leah appeared to have difficulty solving the problem. The low resolution of the listing processes was navigated by periods of time to reflect on the list—both Hallie and Leah were able to read which outcomes had been listed and create in-the-moment strategies.

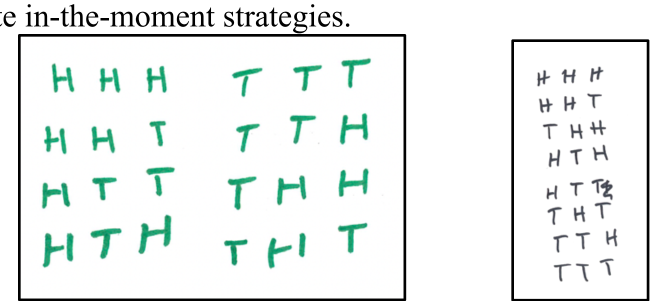

Figure 1: (A) Hallie's list of three coins, (B) Leah's list of three coins

Hallie and Leah also used lists of outcomes to determine the number of ways to flip four and five coins. Leah persevered in her original strategy of ordering the coins by the number of heads, and her list for four coins is shown in Figure 2 (A). While the broader strategy for ordering the outcomes is clear, her strategy within each of the parts appears to have been determined in the moment. She describes how she listed all outcomes with three heads as "I put all the three heads at the start, and then I put them all at the end, and then I put one on the left, and then two on the right, two on the left and one on the right, and that got rid of... this one [crossing out HHHT] was just a repeat of what I did up here." This description shows that she produced these outcomes in an organized way (rotation around a central axis), but notably she chose a different strategy to arrange the outcomes with three tails, and this strategy was not extended for outcomes with two heads and two tails. I contend that her approach to listing the outcomes showed increasing resolution, but each part was structured using in-the-moment techniques. However, the list was a sufficient tool to enact this approach as it allowed her to keep track of her reasoning and reflect on the current state of the list to determine which outcomes had not been counted. I asked Hallie and Leah to find the number of outcomes in each part of their list, and they found that the number of outcomes was $1 + 4 + 6 + 4 + 1 = 16$.

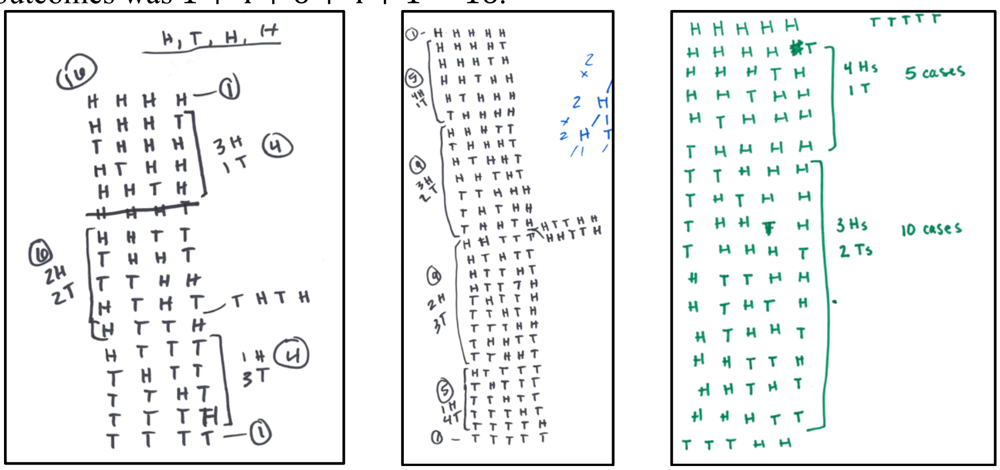

Figure 2. (A) Leah's list for 4 coins (B) Leah's list for 5 coins (C) Hallie's list for 5 coins

Both Hallie and Leah demonstrated increased resolution for five coins, as shown in Figures 2 (B) and (C). Leah's list shows repeated reasoning about ordering for outcomes with four heads and one tail, and outcomes with one head and four tails; however, her counting process for three heads and two tails, and two heads and three tails, demonstrates that the order was still determined in the moment (only after comparison with Hallie's list did she find that she missed two outcomes). Hallie relied on the symmetry of the outcomes in her list, and further she appears to use complete resolution in her listing process. Although the broad structure of her list was identical to Leah's, her organization within each part was consistent throughout. After writing half of the list, she concluded and noted "I kind of stopped writing, because of the symmetry thing we talked about." That is, Hallie reflected on her observations from prior lists and observed that by creating only half the list the remaining outcomes could be produced in the same manner. Hence, her solution reflects the mathematical expression $2(1 + 5 + 10)$. These examples demonstrate how Hallie and Leah used lists as tools for reflection, allowing them to reason about the structure of the list at hand and connecting that structure with cardinality even when using low-resolution listing processes.

Partial Lists for Part of the Problem

In the third interview Hallie and Leah worked on the License Plate Problem, which asked *A license plate consists of an arrangement of six letters and/or numbers. How many license plates consist of three numbers followed by three letters?* For this problem, Hallie and Leah were able to determine that the solution was $10^3 \cdot 26^3$ without using external representations. A subsequent part of the problem asked *How many license plates have four letters and two* *numbers?* The solution to this problem is $\binom{6}{2} \cdot 10^2 \cdot 26^4$, which can be found by selecting the two positions in which the numbers occur, and then arranging the numbers and letters. At this point in the interviews Hallie and Leah appeared adept at solving problems of arrangement with repetition, but this new problem also involved selecting which of the six characters would be letters or numbers as one step in the problem. Hallie and Leah recognized that the solution was $10^2 \cdot 26^4$ multiplied by the number of ways to select two of the six characters, but they were not sure how to determine that number. Hence, they were able to solve one step of the problem but not the other. Their solution to the problem involved listing some of the ways to arrange two numbers and four letters, as shown in Figure 3. This use of listing is both partial in that it omits some ways to arrange numbers and letters, but also each inscription in the list denotes a number of outcomes rather than individual outcomes. Hence, the list was used as a tool for part of the problem, allowing them to physically outsource their reasoning for the parts of the problem they could not perform mentally. Additionally, this example demonstrates how the students began to move away from lists as outcome-oriented tools to process-oriented tools as each 'outcome' in the list stood for a collection of outcomes rather than an individual outcome. Hence, the list itself reflects one stage in the counting process.

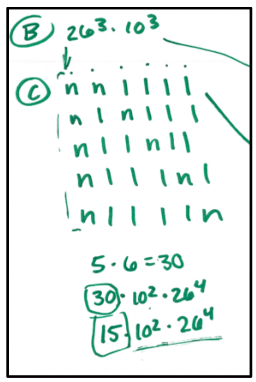

Figure 3: Partial listing of arrangements of two numbers and four letters

Representations of Listing Processes and Connections to Solutions

During the fifth interview the students solved the Three Card Problem, which states *You and your two friends, Ben and Charlene, are playing a card game that only uses spade cards (i.e., your deck consists of the Ace through King of spades). You each draw a card, and whoever has the highest card wins. How many ways are there to play this game so that you come in first, Charlene comes in second, and Ben comes in last?* The solution to this problem is $\binom{13}{3}$, which can be found by determining which three cards will be given to the players, and then giving those cards to the players in a way that preserves win order. Early in the solution Hallie stated "I could just write the cards themselves, since I know who's coming in first, who's coming in second, and who's coming in third," which indicates that she understood that finding the number of ways to select three cards would solve the problem. However, the students had not found a formula for counting selections.

Hallie and Leah employed a listing process that fixed the winning card, iterated through the second card, and then iterated through each losing card for each second card, as shown in Figure 4 (A). Hallie then created a representation of this listing process in Figure 4 (B), which demonstrates all instances where the winning card is the King. The listing process represented in these figures demonstrate complete resolution, showing how Hallie and Leah's application of listing processes was refined over the sequence of interviews. After representing this partial listing process and observing there are $11 + 10 + \cdots + 2 + 1$ outcomes where the winning card is the King, the students then continued the process to find that there were $(11 + 10 + \cdots + 1) + (10 + \cdots + 1) + \cdots + (2 + 1) + (1)$ total outcomes, which reflects the listing process they used. To simplify this expression, Leah observed that the 11 appears in 1 of the sums, the 10 appears in 2 of the sums, and this pattern continued, as demonstrated in Figure 4 (C). I contend that he representations shown in Figure 4 (A) and (B) represent a listing process rather than simply a list of outcomes, as demonstrated by the use of arrows to signify the continuation of a process and brackets accompanied with numbers to signify the numerical value of terminating that process. Hence, it appears that the students simplified their use of notation to accommodate sets with larger cardinality by moving from notations of individual outcomes to notations for processes that generate those outcomes. In addition, their solution to the problem, $11 + 2(10) + 3(9) + \cdots + 10(1)$ reflects their use of the representations, leading to a form that differs substantially from a canonical solution of $\binom{13}{3} = \frac{13!}{3!10!}$.

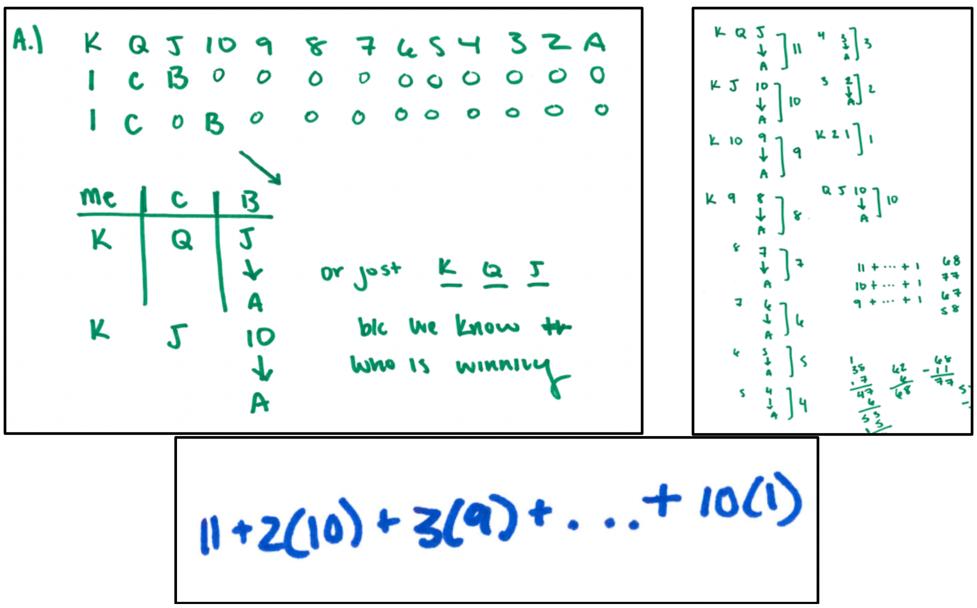

Figure 4: (A) Hallie's articulation of a listing process (B) Hallie's carrying out (C) Leah's solution

Discussion

These data demonstrate how lists of outcomes and listing processes can play a role in student understanding of counting by influencing how students approach a solution, and the form of the solution they find. This was true even for counting problems where the cardinality of the set of outcomes made a complete list intractable to produce by hand. Rather than abandoning lists as a viable strategy, Hallie and Leah adapted their listing techniques by leveraging structural aspects of the set of outcomes (such as symmetry), relying on lists to answer only parts of problems, and representing listing processes rather than lists of outcomes to articulate strategy without needing to carry out that strategy in its entirety. These data inform broader questions of how students create, use, and adapt representations as they solve counting problems because they show how students can become adept at leveraging a specific type of solution approach and become successful in applying that approach to increasingly sophisticated problems.

Acknowledgments

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