Exploring Students' Reasoning With Function Composition Using Graphical Representations

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In this study, we explore how students reasoned on function composition tasks with and without a dynamic graphical representation. Fifteen graduate students individually participated in the task-based and semi-structured interviews to sketch composite functions given their graphs, equations and dynamic graphical representations. One student's responses were analyzed in this paper. Preliminary results suggest that: (1) incorporating multiple reasoning strategies, especially graphical reasoning, and (2) utilizing the dynamic graphical representation, helped build the student's conceptual understanding of function composition.

Keywords: function composition; graphical reasoning; dynamic graphical representations

Starting from secondary school (cf., CCSSM, 2010), students are exposed to, and taught about, function composition via algebraic substitution. Yet, such procedural approaches are limited in their ability to develop sufficiently deep mathematical understandings of function composition, and sometimes even have detrimental effects (e.g., Ayers et al., 1988; Moore & Bowling, 2008; Chen et al., 2023; LaPlace et al., 2024). In this study, we try to explore the ways students reason about the composition of two functions through graphical and other lenses.

Research Question, Background, and Framework

Specifically, we seek to answer the following research question: *How might students reason about function composition with and without the dynamic graphical representation?*

Function Composition in Extant Literature

As the primary operation in function arithmetic, *function composition* [$g \circ f = g\big(f(x)\big)$] is a key content topic in algebra through calculus that helps undergraduate students learn about functions effectively (Ratliff & Garofalo, 2006; LaRue & Infante, 2015). Previous research emphasized the importance of functions and function composition in advanced mathematical topics, such as the chain rule which was described by Webster (1978) as "crucial" and "fundamental" in calculus and its applications (e.g., Abramson, 2021; Clark et al., 1997; Gordon, 2005). However, limited studies have focused on function composition. Studies that have explored students' meaning for function composition across algebraic rules, tables, and graphs largely observe an overreliance on algebraic procedures when engaging with function composition (Bowling, 2009; Engelke et al., 2005; Miller et al. 2015), while also recording the limitations of algebraic approaches (Chen et al., 2023; Meel, 2003; LaPlace et al., 2024).

Dreyfus and Eisenberg (1983) pointed out that without a formula, the graphical representation of function has very little meaning for most students entering calculus. In recent years, increasing numbers of studies have argued that multiple representations might contribute to better conceptualization of function and related topics (i.e. composition), and suggested more attention to graphical perspectives including animated graphs (e.g., Ainsworth, 1999; Artigue, 1992; Chen et al., 2023; LaPlace et al., 2024; Parr, 2023; Van & White, 2004). Furthermore, researchers have claimed that the graphical interpretation would be beneficial for students' understanding of function composition (in chain rule and other calculus topics) (Gordon, 2005; LaRue & Infante, 2015), yet studies have not explored in what ways students might reason about function composition other than algebraic substitution. In conclusion, there is a dearth of literature on function composition, especially through a graphical lens.

Graphical and Algebraic reasoning

Educators in secondary and college mathematics courses typically approach function composition through algebraic equations $(g \circ f)(x) = g\big(f(x)\big)$. However, traditional instruction through the algebraic substitution of equations does not provide much insight about composing functions when students are provided with graphical representations. Graphs tend to make more visible some properties of functions, such as domain, range, symmetry and continuity. For this study, we differentiate reasoning on equation-based characteristics of functions, which is related to algebraic reasoning (Cuoco et al., 1996; Kriegler, 2008), from reasoning based on graphical characteristics of functions, which is likely similar to geometric reasoning (Crowley, 1987; Cuoco et al., 1996). To help exemplify this difference, a function, algebraically, is often characterized by its equation, e.g., $f(x) = 2x + 1$ (i.e., the structure of the set of points included in the relation); whereas, graphically, we depict a function with a graph (typically a geometric curve in the Euclidean plane), e.g., $f(x)$ is a line.

Methods

To explore how students reason about function composition through graphical and other lenses, with and without dynamic graphical representation, this study employed a qualitative multiple case study through face-to-face and one-on-one task-based and semi-structured interviews between the researcher and participants. As a desirable sample of this study to collect rich and informative data, it would be preferable for the participants to have basic understanding of function composition to be able to reason during the interview. Therefore, university graduate students (n=15) chosen from purposive sampling in a mathematics education program were interviewed.

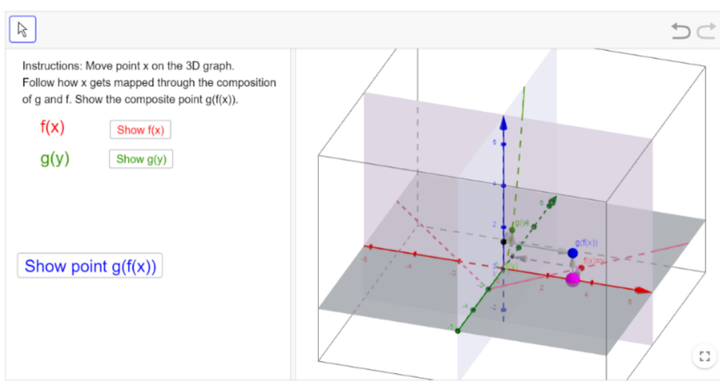


Figure 1. Dynamic Graphical Representation in GeoGebra

The dynamic graphical representation in this study (displayed in Figure 1) is a three-dimensional simulation applet of function composition in GeoGebra. The set up demonstrates a point-by-point analysis of function composition: the user can drag the pink point along the $x - axis$, while the corresponding value of $f(x)$ (absolute value function in red in Figure 1 as an example) will be plotted on the $y - axis$ by an arrow simultaneously; this point is also the input of the function $g(y)$ (where $y = f(x)$). Meanwhile, the value of $g(f(x))$ will be shown on the $z - axis$. Coordinating this point's movement along with the movement of the point on the $x - axis$ (the blue dot in Figure 1) produces the composite function. I aim to explore the affordance of this three-dimensional dynamic graphic representation on students' reasoning, and to study the possible learning opportunities as well as obstacles when using this representation on solving function composition problems.

The task-based interview contained three rounds. The applet was introduced to participants after the first two rounds, and before the third round where students are asked to use the dynamic representations. Each round of the interview consisted of four function composition problems (due to space constraints, one problem is displayed as an example in Figure 2), and participants were asked to sketch $g(f(x))$ functions given static graphs (Round 1), equations (Round 2) and dynamic graphical representations as well as static graphs (Round 3). There were some shared problems across the rounds in order to make students' responses comparable across the three rounds of interviews. In each round, the researcher asked participants to think aloud while sketching, and students were interviewed at the end of each round with follow-up questions. Interviews were recorded, and video as well as audio data were transcribed. The qualitative data sources included participants' written responses, students' actions when solving the problems, and the transcription of their oral responses.

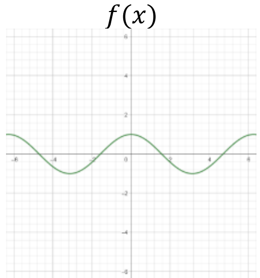
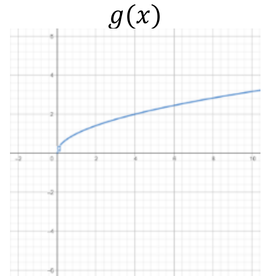
$f(x)$ 	$g(x)$ 	$f(x) = \cos(x)$ $g(x) = \sqrt{x}$

Figure 2. Example: Round 2 Problem 3 (equations only given) and Round 3 Problem 4 (graphs only given)

After initially reading through the interview data, I focused on one participant, Jade (pseudonym), who exemplified rich and interesting strategies of graphical reasoning and other reasoning patterns. The analysis concentrated on identifying the approaches and patterns of reasoning across the problems as well as describing any distinct ways of reasoning graphically in each problem. For the preliminary stage, this study adapted the themes presented in Chen et al., (2023) for the ways students reason function composition. Participant's written and oral responses were coded into categories of: (1) equation related *algebraic reasoning*; (2) graph related *graphical reasoning*; and (3) reasoning through *discrete points*. These were then listed as reasoning steps in chronological order. For example, the participant responded: "I plot some points, 2, 4, 8…and I find it (the graph of $g(f(x))$) grows slower than this one (pointed at the graph $g(x)$), so the graph (pointed at the $g(f(x))$ graph) is like this." This statement above was coded to a step of *discrete points* followed by a step of *graphical reasoning*. In addition, the detailed properties of graphs in the participant's graphical reasoning steps were also identified.

Preliminary Findings

In general, Jade successfully sketched 10 out of 12 composite function graphs. This was an extremely high success rate, especially given results from prior students (e.g., LaPlace et al., 2024). Figure 3 provides an overarching picture of the chronological solving process Jade went through for each problem in each of the three rounds.

Figure 3. Jade's solving process with reasoning steps (algebraic reasoning in yellow, graphical reasoning in blue and discrete points in green)

In most problems (11 out of 12), Jade applied more than one way of reasoning, and half of the problems (6 out of 12) were solved utilizing all three types of reasoning. Figure 3 reveals that, for Jade, different reasoning approaches in one problem were not isolated from each other, instead, the interplay among ways of reasoning drove the solving process forward. In particular, the pattern of alternately using graphical approaches and discrete points is typical (e.g., Round 1 Problem 3). In this pattern, Jade usually repeatedly applied the strategy of: (1) observe certain properties of the general function shape, and then (2) examine her estimation or specify the precise location of the shape by checking specific points on the function graphs.

Jade was an interesting case because graphical reasoning was a leading approach across three rounds (63%, 68 out of in total 108 steps), especially in round 1 and 3 when f and g were given initially as graphs (not equations). It is not surprising that Jade approached problems through algebraic thinking more frequently in round 2 when given equations. Even so, graphical reasoning still appeared in parts of those problems. Jade applied graphical reasoning through the general shape, domain and range, continuity, symmetry, concavity, and periodicity of function graphs. For instance, for domain and range in one problem Jade reasoned: "The first thing I know is because $x \geq 0$ (pointed at $\mathrm{g}(x)$), so we only pick the point above x-axis (pointed at and meant the output of $f(x)$)…" This graphical observation critical for informing for which *x* the composite function is defined.

Unlike existing literature (e.g., Miller et al., 2015), Jade did not show an overreliance on algebraic procedures. From Figure 3 it can be concluded that her attempts at reasoning algebraically mostly happened at the beginning of the solving process, and then Jade paid more attention to the other two reasoning strategies. This is probably because – particularly with the intentionally unusual and unfamiliar combinations of parent functions in many of the problems – the equation of the composite function is still difficult to use to quickly generate its graph. We do note, however, that Jade incorrectly sketched the composite function to be the line $y = x$ without considering the domain issue, when she algebraically figured $f(x) = \ln(x)$ and $g(x) = e^x$ (Round 2 Problem 1) simply "cancelled out with each other". This mistake coincides with what the literature raised about the negative impact of relying solely on an algebraic approach (e.g., Chen et al., 2023).

Compared with the rounds without dynamic graphical representations, Jade solved the tasks with fewer reasoning steps in round 3 when she was supported with the dynamic graphical representation (see Figure 3). Moreover, the fewest number of algebraic steps appeared in round 3 as Jade only tried to reason from equations once. The dynamic representation supported her reasoning about function composition without the aid of equations. Jade also developed more sophisticated conceptual understanding of function composition while interacting with the dynamic graphical representations. In Round 2 Problem 1, Jade could not graphically explain why two curves ($f(x) = \ln(x)$ and $g(x) = e^x$) combined to a straight line ($y = x$ on the positive domain). By comparison, with the same problem for Round 3 Problem 1, using the dynamic graphic representation Jade concluded and explained the composition of inverse function will make a line of $y = x$ because: "…when the blue ball moves (point of $g(f(x))$), the pink (point of initial input x) and green ball (point of $g(y)$) moves with the same pace". At the end of Round 3, Jade shared: "…It's like (the dynamic representation is) more direct for me to find (how) the x of $f(x)$ and the y for $g(y)$ make the final composition of the function." From the experience with the dynamic graphical representation, she changed her opinion from previous rounds and concluded that "…for the shape of the final graph, I think we need to consider both two (parent function) graphs."

Discussion and Future Research

This novel study investigated patterns in the way a student reasoned about function composition when given static and dynamic graphical representations. It reported, for the first time, a student's detailed graphical reasoning approaches applied on composing two functions.

The results of Jade's case suggest the benefits of incorporating diverse reasoning approaches. Particularly, one prominent and effective strategy Jade implemented was approaching properties of function graphically, and alternately confirming or refining the sketch of a composite function's graph with discrete points. This finding demonstrates the feasibility and the advantages of understanding function composition beyond algebraic substitution.

Jade's responses also revealed the promising potential of graphical reasoning, as it is fully possible to sketch the composite function graph by reasoning exclusively through graphical lenses without knowing or using the function equations. Jade's reasoning path in Round 3 Problem 4 is one example, where even with her previous experiences with the equations in Round 2 Problem 3, Jade still addressed this problem solely through graphical approaches.

Lastly, yet importantly, the comparison between Jade's reasoning with and without the dynamic graphical representation highlighted the possibility of using this representational form as a viable tool. Almost no algebraic reasoning approach appeared in round 3, which indicates the potential of the dynamic graphical representation to foster graphical reasoning, and to support students in overcoming an overreliance on algebraic equations. More significantly, with the dynamic graphical representation Jade showed a higher-level of understanding on the behavior of function graphs and how the graphs interact with each other when being composed.

I am interested to learn from audience: (1) how can college instructors introduce function composition through a graphical lens? (2) how do you think this approach can benefit or hinder students' learning about advanced calculus related topics, such as the chain rule for derivatives?

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