Coordinate Representation of Vectors and Matrix Representations of Linear Transformations in a DGS Assisted Linear Algebra Learning Environment: The Case of Polynomial Vector Space

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The contributed report highlights the understanding of matrix representations of linear transformations mapping finite-dimensional polynomial vector spaces among university mathematics majors within a dynamic geometry software (DGS)-assisted linear algebra learning environment. Through an analysis of data from in-depth qualitative interviews, framed by the theories of representations in linear algebra (Dorier & Sierpinska, 2001; Hillel, 2000) and the concreteness principle (Harel, 2000), the findings reveal a range of innovative and creative ways in which the students integrated visual and analytical approaches to understand, produce, and interpret matrix representations of linear transformations. The results contribute new insights into the pedagogy of university-level linear algebra, highlighting the growing understanding of the role of visualization in the teaching and learning of the subject. The study also adds to the increasing body of evidence supporting the importance of visualization in linear algebra education.

Keywords: matrix representation; linear transformation; polynomial vector space; differential operator; dynamic geometry software

Researchers widely focused on the pedagogy of linear algebra (Dogan-Dunlap, 2010; Dorier, 1991, 1995, 1998; Dreyfus, Hillel & Sierpinska, 1998; Gueudet-Chartier, 2006; Harel, 1987, 1989, 1990; Parraguez & Oktac, 2010; Pavlopoulou, 1993; Robert & Robinet, 1989; Rogalski, 1994; Sierpinska, 1995, 2000; Sierpinska, Dreyfus & Hillel, 1999; Sinclair & Gol Tabaghi, 2010; Thomas & Stewart, 2011; Zandieh, Wawro & Rasmussen, 2017) along with representations in R^n and the multi-representational aspect of linear algebra (Dorier, Sierpinska, 2001; Hillel, 1997, 2000). Harel and Kaput (1991), and Harel (1990) formulated the concreteness principle, as a fundamental approach for the teaching and learning of linear algebra. He recommends MATLAB as a tool that would help students visualize vectors and matrices as concrete mathematical objects, in accordance with the concreteness principle. Aligned with the concreteness principle (Harel, 2000) and in search for a desirable matrix representation for a linear transformation as brought forward in Hillel (2000), this study focuses on linear algebra students' understandings of the matrix representations of linear transformations mapping finite dimensional polynomial vector spaces in a dynamic environment.

Representations in Linear Algebra

Describing linear algebra as an explosive compound of systems of representation, Dorier and Sierpinska (2001) classifies the representations used in linear algebra as the graphical, the tabular, and the symbolic representations (p. 270). Hillel (2000) views basis-dependent representations in linear algebra as the major cause of difficulties for students of introductory linear algebra courses. Basis-dependent representations are not unique; these representations are contingent upon the particular basis or bases used. The two major challenges for linear algebra students are the understanding of such basis-dependent structures along with the understanding of representational invariants, that is, the understanding of what stays invariant irrespective of the bases that are being employed in representing linear algebraic structures (p.199). In general, when representing linear transformations like matrices, students often have difficulties in interpreting the columns of these matrix representations. A matrix representation itself is used to represent a linear transformation, yet, students are tempted to think of columns of this matrix as the images of the corresponding basis vectors under the linear transformation: “It stems from thinking that the columns of the matrix representation of an operator are, so to speak, images of the basis vectors, not representations of these images relative to a basis” (p.203-204). In their exploration of linear transformations, the mathematics majors in this study creatively and innovatively integrated visual and analytic approaches to construct meaningful matrix representations. Thus, the study is framed within the guiding theoretical perspectives of the concreteness principle (Harel, 2000) and the multi-representational context of linear algebra (Dorier & Sierpinska, 2001; Hillel, 2000).

Table 1. Interview tasks: Find the matrix A' for $T: \mathbb{P}_n \to \mathbb{P}_m$ relative to the bases B and B'.

Task 1	♦Multiplication by x Operator $\begin{cases} T: \mathbb{P}_2 \to \mathbb{P}_3 \\ p \mapsto xp \end{cases}$	$\mathcal{B} = \{1, x, x^2\}; \mathcal{B}' = \{1, x, x^2, x^3\}$
Task 2	♦Multiplication by x^2 Operator $\begin{cases} T: \mathbb{P}_2 \to \mathbb{P}_3 \\ p \mapsto x^2 p \end{cases}$	$\mathcal{B} = \{1, x, x^2\}; \mathcal{B}' = \{1, x, x^2, x^3, x^4\}$
Task 3	♦Differential Operator $\begin{cases} D_x: \mathbb{P}_2 \longrightarrow \mathbb{P}_1 \\ p \mapsto p' \end{cases}$	$\mathcal{B} = \{1, x, x^2\}; \mathcal{B}' = \{1, x\}$
		$\mathcal{B} = \{1 + x, x - x^2, x + x^2\}; \mathcal{B}' = \{1 - x, 1 + x\}$
Task 4	♦Differential Operator $D_x: \mathbb{P}_3 \longrightarrow \mathbb{P}_2$	$\mathcal{B} = \{1 + x^2, x - x^3, 1 - x^2, x + x^3\}; \mathcal{B}' = \{1 + x, x - x^2, x + x^2\}$
Task 5	♦Integral operator $\begin{cases} J: \mathbb{P}_2 \longrightarrow \mathbb{P}_3 \\ x^k \longmapsto \int_0^x t^k dt \end{cases}$	$\mathcal{B} = \{1 - x, x - x^2, 1 - x + x^2\}; \mathcal{B}' = \{1, x, x^2, x^3\}$
		$\mathcal{B} = \{1 + x, x - x^2, x + x^2\}; \mathcal{B}' = \{1, x, x^2, x^3\}$
		$\mathcal{B} = \{1, x, x^2\}; \mathcal{B}' = \{1 + x^2, x - x^3, 1 - x^2, x + x^3\}$
Task 6	♦Integral operator $\begin{cases} J: \mathbb{P}_3 \longrightarrow \mathbb{P}_4 \\ x^k \longmapsto \int_0^x t^k dt \end{cases}$	$\mathcal{B} = \{1, x, x^2, x^3\}; \mathcal{B}' = \{1, x, x^2, x^3, x^4\}$
		$\mathcal{B} = \{1 + x, x - x^2, x + x^2, x^3\}; \mathcal{B}' = \{1, x, x^2, x^3, x^4\}$

Sample Interview Outline: Is T/D/J a linear transformation? What does it mean for T/D/J to be a linear transformation? What is the standard matrix representation? Is it possible to have a square matrix as the matrix representation of T/D/J? Is it possible to use non-standard bases in the expression of the matrix representation of T/D/J? How would you multiply/differentiate/integrate a polynomial using the matrix representation of T/D/J? How would you compare the two approaches? How does the matrix representation you proposed act on the polynomial on the DGS ? How would you distinguish between the actual object from its representation?

Context and Method

Qualitative-descriptive interview data were collected at a U.S. university as part of a research project aimed at enhancing math and math education majors' understanding of advanced mathematics, with a particular focus on the connections between geometry and linear algebra using DGS-MATLAB technology. The qualitative interviews followed a semi-structured interview model (Kvale, 2007), where the interviewer used follow-up probes and questions to delve deeper into the interviewees' responses. Sixteen math majors participated and answered a series of questions centered around matrix representations (see Table 1). The tasks were designed to be explored through both visual and analytic approaches, in line with the study's theoretical framework. Although optional, the math majors were eager and enthusiastic about using MATLAB/DGS, primarily to check their work, visualize analytic solutions, test conjectures, and provide examples or counterexamples.

The analysis of the videotaped interviews, along with MATLAB-DGS inscriptions, was conducted using thematic analysis (Boyatzis, 1998) and the VA (visual-analytic) strategy coordination method (Zazkis, Dubinsky, & Dautermann, 1996). This methodology was employed to describe the innovative and creative ways in which participants coordinated visual and analytic approaches to make sense of matrix representations of linear transformations. After transcribing the interview videos, the author reviewed the videos and original transcripts line by line, using thematic analysis in coordination with the interview analysis method described by Zazkis et al. (1996), to identify the strategies (visual and analytic) employed by students in exploring matrix representations. The final stage of data analysis involved a comprehensive review of the entire dataset multiple times, following the constant comparative methodology (Glaser & Strauss, 1967).

Results

This section considers the polynomial multiplication operator (Tasks 1-2), the differential operator (Tasks 3-4), and the integral operator (Tasks 5-6). In each task, students were asked to find the matrix representation A' for the given linear transformation T relative to bases $\mathcal{B}$ and $\mathcal{B}'$. The analysis that follows focuses primarily on student-generated approaches and strategies.

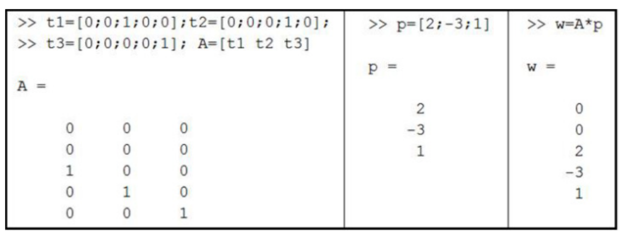

Figure 1. S1's analytic-arithmetic representation for Task 2.

The Polynomial Multiplication Operator

Most of the research participants thought of Tasks 1-2 illustrating standard matrices of linear transformations that act as a polynomial-multiplication operator as straightforward and sufficed with the analytic representation by expressing $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)$ in terms of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ accompanied by the MATLAB verification process in obtaining the matrix representations. S1, for instance, first expressed the image vectors $T(1) = x, T(x) = x^2, T(x^2) = x^3$ in terms of the vectors of the standard basis $\mathcal{B}'$ of the codomain vector space $\mathbb{P}_3$ as $T(1) = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3 + 0\mathbf{w}_4, T(x) = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3 + 0\mathbf{w}_4, T(x^2) = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 0\mathbf{w}_3 + 1\mathbf{w}_4$. She then wrote the 4×1 coordinate matrices $[T(\mathbf{e}_1)]_{\mathcal{B}'}, [T(\mathbf{e}_2)]_{\mathcal{B}'}, [T(\mathbf{e}_3)]_{\mathcal{B}'}$ using the coefficients in each linear combination in the previous step. Upon juxtaposing these coordinate matrices she obtained the desired 3×4 matrix $A = [[T(\mathbf{e}_1)]_{\mathcal{B}'} \quad [T(\mathbf{e}_2)]_{\mathcal{B}'} \quad [T(\mathbf{e}_3)]_{\mathcal{B}'}]$. To illustrate how the multiplication-by-x-operator works via the matrix representation, S1 first converted the polynomial $\mathbf{p} = 2 - 3x + x^2$ to its column vector representation $[\mathbf{p}]_{\mathcal{B}} = [2 \quad -3 \quad 1]^T$ relative to the standard basis $\mathcal{B}$ of the domain vector space $\mathbb{P}_2$. Upon applying the matrix A on $[\mathbf{p}]_{\mathcal{B}}$, she obtained $A[\mathbf{p}]_{\mathcal{B}} = [T(\mathbf{p})]_{\mathcal{B}'}$. Finally, she retrieved the image polynomial $T(\mathbf{p})$ from its coordinate representation $[T(\mathbf{p})]_{\mathcal{B}'}$ via the linear combination expression $T(\mathbf{p}) = 0\mathbf{w}_1 + 2\mathbf{w}_2 - 3\mathbf{w}_3 + 1\mathbf{w}_4 = 2x - 3x^2 + x^3$. S1's approach for the other warm-up example was similar, as illustrated in Figure 1.

The Elementary Differentiation Operator

Interviews on differential operators (Tasks 3-4) began with the warm-up activity in which math majors explored the standard matrix of the differential operator $D_x: \mathbb{P}_2 \longrightarrow \mathbb{P}_1$. The common approach was the analytic one: (i) Math majors first determined the 3 image vectors $D_x(1), D_x(x), D_x(x^2)$ by applying the linear transformation D_x to each vector of the standard basis $\mathcal{B} = \{\mathbf{e}_1 = 1, \mathbf{e}_2 = x, \mathbf{e}_3 = x^2\}$ of the domain vector space $\mathbb{P}_2$; (ii) They then expressed each image vector $D_x(1) = 0, D_x(x) = 1, D_x(x^2) = 2x$ in terms of the vectors of the basis $\mathcal{B}' = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x\}$ of the codomain vector space $\mathbb{P}_1$ as $D_x(1) = 0\mathbf{w}_1 + 0\mathbf{w}_2, D_x(x) = 1\mathbf{w}_1 + 0\mathbf{w}_2, D_x(x^2) = 0\mathbf{w}_1 + 2\mathbf{w}_2$; (iii) They then wrote three 2×1 coordinate matrices $[D_x(\mathbf{e}_1)]_{\mathcal{B}'} = [0 \quad 0]^T, [D_x(\mathbf{e}_2)]_{\mathcal{B}'} = [1 \quad 0]^T, [D_x(\mathbf{e}_3)]_{\mathcal{B}'} = [0 \quad 2]^T$ using the coefficients in each linear combination in the previous step; (iv) Finally, they combined the coordinate matrices from the previous step to form the sought 2×3 matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Students then switched to the MATLAB approach to visualize what has been analyzed, and to use the 2×3 matrix A to evaluate $D_x(\mathbf{v})$ where $\mathbf{v} = 2 - 3x + x^2$. For that purpose, the common approach was to enter the 2×3 matrix A and the vector $\mathbf{v}$ by typing **D1=[0;0]; D2=[1;0]; D3=[0;2]; A=[D1 D2 D3]** and **v=[2;-3;5]**, respectively. Finally, typing **w=A*v** resulted in a column vector with components -3 and 10 (fig.2a). S2, for instance, stopped at this stage and remarked that the vector's components were the same as the coordinates because she was dealing with standard basis representations. Although several students stopped at this stage; some others further indicated that $[-3 \quad 10]^T$ was the coordinates of the vector sought, and that the additional step of moving back from coordinate vector representation $[-3 \quad 10]^T$ to the vector itself was necessary, whereby deduced the answer as $\mathbf{w} = -3 + 10x \in \mathbb{P}_1$. Students who felt the need to take this additional step of mapping the coordinate representation back to the actual vector itself could be thought of as being aware of the distinction between the components of a vector and the coordinates relative to the standard basis, even though the same set of numbers were used to express them (Hillel, 2000, p.201).

In another task, students were asked to determine the matrix representation of the differential operator $D_x: \mathbb{P}_3 \longrightarrow \mathbb{P}_2$ using the nonstandard bases $\mathcal{B} = \{\mathbf{v}_1 = 1 + x^2, \mathbf{v}_2 = x - x^3, \mathbf{v}_3 = 1 - x^2, \mathbf{v}_4 = x + x^3\}$ and $\mathcal{B}' = \{\mathbf{w}_1 = 1 + x, \mathbf{w}_2 = x - x^2, \mathbf{w}_3 = x + x^2\}$. Unlike the previous task in which students had begun with the analytic approach, in this task all students began with a visual approach primarily for the purpose of verifying that the given sets $\mathcal{B}$ and $\mathcal{B}'$ were indeed bases for $\mathbb{P}_3$ and $\mathbb{P}_2$, respectively (fig.2b-e). One group of students verified this via the traditional rref procedure on MATLAB (fig.2b,d). Those who embraced the DGS approach created sliders in an attempt to demonstrate that any polynomial of degree ≤ 3 (or degree ≤ 2) could be written as a linear combination of the basis vectors of the set $\mathcal{B}$ (or $\mathcal{B}'$), respectively (fig.2c,e). Upon establishing $\mathcal{B}$ and $\mathcal{B}'$ as bases, students switched to the analytic procedure in order to obtain the matrix representation of the differential operator. S3 expressed each image vector $D_x(1 + x^2) = 2x, D_x(x - x^3) = 1 - 3x^2, D_x(1 - x^2) = -2x, D_x(x + x^3) = 1 + 3x^2$ in terms of the vectors of the basis $\mathcal{B}' = \{\mathbf{w}_1 = 1 + x, \mathbf{w}_2 = x - x^2, \mathbf{w}_3 = x + x^2\}$ of the codomain vector space $\mathbb{P}_2$ as $D_x(\mathbf{v}_1) = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 1\mathbf{w}_3, D_x(\mathbf{v}_2) = 1\mathbf{w}_1 + 1\mathbf{w}_2 - 2\mathbf{w}_3, D_x(\mathbf{v}_3) = 0\mathbf{w}_1 - \mathbf{w}_2 - \mathbf{w}_3, D_x(\mathbf{v}_4) = 1\mathbf{w}_1 - 2\mathbf{w}_2 + 1\mathbf{w}_3$. In the third stage, S3 wrote the corresponding 3×1 coordinate matrices $[D_x(\mathbf{v}_1)]_{\mathcal{B}'} = [0 \quad 1 \quad 1]^T, [D_x(\mathbf{v}_2)]_{\mathcal{B}'} = [1 \quad 1 \quad -2]^T, [D_x(\mathbf{v}_3)]_{\mathcal{B}'} = [0 \quad -1 \quad -1]^T, [D_x(\mathbf{v}_4)]_{\mathcal{B}'} = [1 \quad -2 \quad 1]^T$ using the previously determined coefficients in each linear combination from the second stage. Finally, he obtained the matrix representation of the differential operator by juxtaposing the four column vectors from the third stage as the 3×4 matrix $A = [[D_x(\mathbf{v}_1)]_{\mathcal{B}'} \quad [D_x(\mathbf{v}_2)]_{\mathcal{B}'} \quad [D_x(\mathbf{v}_3)]_{\mathcal{B}'} \quad [D_x(\mathbf{v}_4)]_{\mathcal{B}'}] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & -2 & -1 & 1 \end{bmatrix}$.

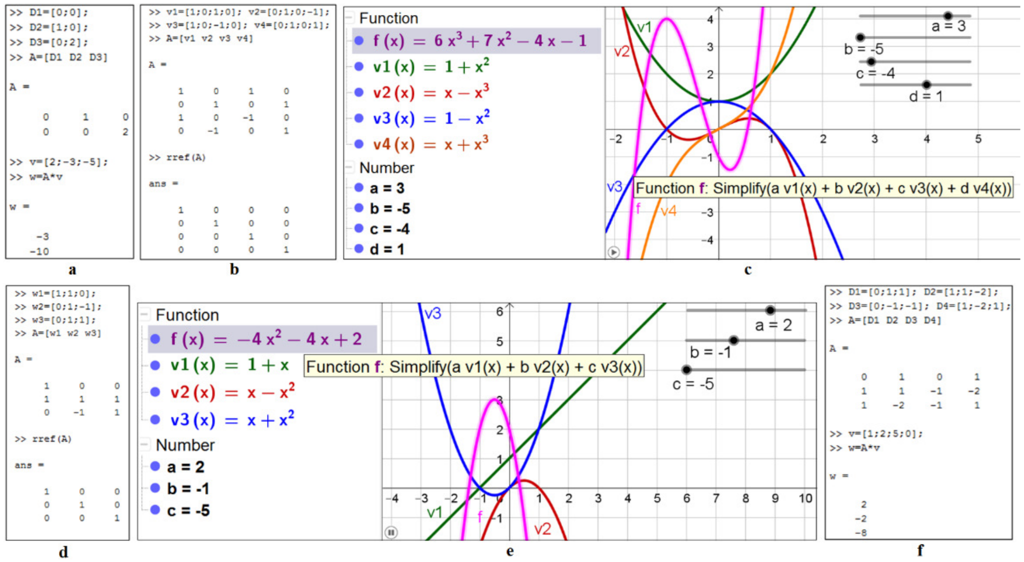

Figure 2. Analytic and visual approaches.

The interviewer then asked the research participants to evaluate the derivative of the cubic polynomial $\mathbf{v} = 6 + 2x - 4x^2 - 2x^3$ via the two approaches and then to compare the two answers. Using the power rule, students easily determined the derivative of the cubic polynomial as $D_x(\mathbf{v}) = 2 - 8x - 6x^2$. In an attempt to see how the matrix approach works, S4 first expressed the cubic polynomial $\mathbf{v}$ in terms of the vectors of the basis $\mathcal{B} = \{\mathbf{v}_1 = 1 + x^2, \mathbf{v}_2 = x - x^3, \mathbf{v}_3 = 1 - x^2, \mathbf{v}_4 = x + x^3\}$ of the domain vector space $\mathbb{P}_3$ simply as $\mathbf{v} = 6 + 2x - 4x^2 - 2x^3 = 1\mathbf{v}_1 + 2\mathbf{v}_2 + 5\mathbf{v}_3 + 0\mathbf{v}_4$; then obtained the coordinate vector representation of the cubic polynomial as $[\mathbf{v}]_{\mathcal{B}} = [1 \quad 2 \quad 5 \quad 0]^T$. She then switched to MATLAB and followed a procedure similar to S2's (fig.2a) and determined the matrix applied coordinate vector $A[\mathbf{v}]_{\mathcal{B}} = [2 \quad -2 \quad -8]^T$ as shown (fig.2f). Upon the interviewer's probing whether she would want to stop or continue, S4 responded that she should continue, explaining that what she had was not the vector itself but its image, specifically its coordinate vector representation according to the basis B prime. She added that she still needed to find the vector itself, or the image polynomial. Using the linear combination analytic approach, she determined the image polynomial as $2\mathbf{w}_1 - 2\mathbf{w}_2 - 8\mathbf{w}_3 = 2(1 + x) - 2(x - x^2) - 8(x + x^2) = 2 - 8x - 6x^2$.

In another task of determining the matrix representation of the differential operator $D_x: \mathbb{P}_2 \longrightarrow \mathbb{P}_1$ using the nonstandard bases $\mathcal{B} = \{\mathbf{v}_1 = 1 + x^2, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2\}$ and $\mathcal{B}' = \{\mathbf{w}_1 = 1 - x, \mathbf{w}_2 = 1 + x\}$, upon establishing the basisness of $\mathcal{B}$ and $\mathcal{B}'$ in a similar manner as in the previous task via DGS-MATLAB approach, S5 first obtained the matrix representation of the differential operator as a 2×3 matrix D via the analytic-arithmetic procedure in a manner similar to S4's, then demonstrated how this matrix representation actually acts on the coordinate representation $[\mathbf{v}]_{\mathcal{B}}$ of the previously explored polynomial $\mathbf{v} = 2 - 3x + 5x^2$ in the DGS. Upon graphing all the key actors (fig.3a-c); S5 concluded her exploration by verifying that the image polynomial (fig.3d) obtained both ways coincide via the linear combination syntax in a similar manner as before.

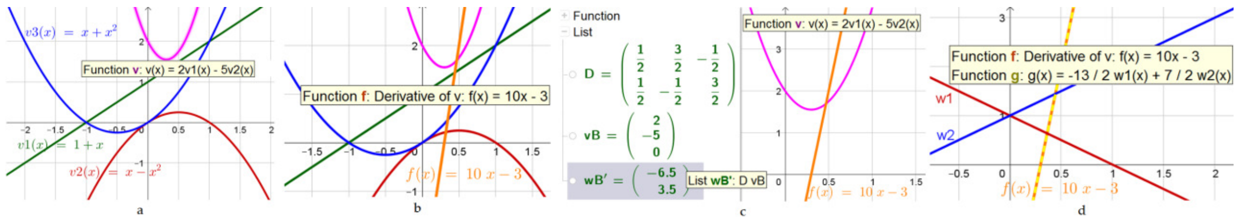

Figure 3. S5's visual approach for Task 3.

The Elementary Integral Operator

In the previous tasks on the elementary differential operator $D_x: \mathbb{P}_n \longrightarrow \mathbb{P}_{n-1}$, math majors made use of both analytic and visual approaches by involving DGS and MATLAB in their explorations. Such a visual-analytic coordination proved to be the case in the investigation of the integral operator mapping polynomials vector spaces as well. In the task of finding the matrix representation of the integral operator $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_3$ defined by the equation $T(x^k) = \int_0^x t^k dt$ using a nonstandard basis $\mathcal{B} = \{\mathbf{v}_1 = 1 + x, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2\}$ for $\mathbb{P}_2$ and the standard basis $\mathcal{B}'$ for $\mathbb{P}_3$, S6 first analytically verified that $\mathcal{B} = \{\mathbf{v}_1 = 1 + x, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2\}$ was actually a basis for $\mathbb{P}_2$ via the straightforward rref procedure. She then applied the analytic procedure via which she obtained the coordinate vector representations $[T(\mathbf{v}_1)]_{\mathcal{B}'} = [0 \quad 1 \quad 1/2 \quad 0], [T(\mathbf{v}_2)]_{\mathcal{B}'} = [0 \quad 0 \quad 1/2 \quad -1/3], [T(\mathbf{v}_3)]_{\mathcal{B}'} = [0 \quad 0 \quad 1/2 \quad 1/3]$; then switched to the MATLAB approach by introducing the three 4×1 coordinate matrices by typing **T1=[0;1;1/2;0]; T2=[0;0;1/2;-1/3]; T3=[0;0;1/2;1/3]**. She then thought aloud and stated that because $\mathcal{B}' = \{1, x, x^2, x^3\}$ for $\mathbb{P}_3$ was a standard basis, she could simply combine the three 4×1 column vectors to form the desired matrix representation via the syntax **A=[T1 T2 T3]**.

To test her visualization, she first obtained the coordinate vector representation $[\mathbf{v}]_{\mathcal{B}} = [1 \quad -1 \quad -1]^T$ of the polynomial $\mathbf{v} = 1 - x$ relative to the basis $\mathcal{B} = \{\mathbf{v}_1 = 1 + x, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2\}$ in the domain vector space $\mathbb{P}_2$. Then she applied the matrix A to $[\mathbf{v}]_{\mathcal{B}} = [1 \quad -1 \quad -1]^T$, resulting in $A[\mathbf{v}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}'} = [0 \quad 1 \quad -1/2 \quad 0]^T$ (fig.4a). She interpreted this as the coordinate vector representation of the polynomial $\mathbf{w} = x - (1/2)x^2$ relative to the standard basis for the codomain vector space $\mathbb{P}_3$. She concluded by indicating that the real things are related to each other via integration, that is, the fact that the integral of $\mathbf{v}$ polynomial would result in the $\mathbf{w}$ polynomial under the T operator. In a similar task of finding the matrix representation of the integral operator $T: \mathbb{P}_3 \longrightarrow \mathbb{P}_4$ defined by the equation $T(x^k) = \int_0^x t^k dt$ using a nonstandard basis $\mathcal{B} = \{\mathbf{v}_1 = 1 + x, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2, \mathbf{v}_4 = x^3\}$ for $\mathbb{P}_3$ and the standard basis $\mathcal{B}' = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x, \mathbf{w}_3 = x^2, \mathbf{w}_4 = x^3, \mathbf{w}_5 = x^4\}$ for $\mathbb{P}_4$, S7 followed almost the same approach as S6 with the slight difference in her concluding steps where she demonstrated the action of the matrix representation on the polynomial $\mathbf{v} = 2 - x + 3x^2 + 4x^3$ with the coordinate vector representation $[\mathbf{v}]_{\mathcal{B}}$ (fig.4b).

```
>> T1=[0;1;1/2;0]; T2=[0;0;1/2;-1/3]; T3=[0;0;1/2;1/3];
>> A=[T1 T2 T3]

A =

         0         0         0
    1.0000         0         0
    0.5000    0.5000    0.5000
         0   -0.3333    0.3333

>> v=[1;-1;-1];
>> w=A*v

w =

         0
    1.0000
   -0.5000
         0
```

```
>> T1=[0;1;1/2;0;0]; T2=[0;0;1/2;-1/3;0];
>> T3=[0;0;1/2;1/3;0]; T4=[0;0;0;0;1/4];
>> A=[T1 T2 T3 T4]

A =

         0         0         0         0
    1.0000         0         0         0
    0.5000    0.5000    0.5000         0
         0   -0.3333    0.3333         0
         0         0         0    0.2500

>> v=[2;-3;0;4];
>> w=A*v
```

```
w =

         0
    2.0000
   -0.5000
    1.0000
    1.0000

>> syms x
>> BPRIME=[1;x;x^2;x^3;x^4]; w=sum(w.*BPRIME)

w =

x^4 + x^3 - x^2/2 + 2*x
```

a b

Figure 4. Students' MATLAB approach for Task 5.

In another version of the same task for finding the matrix representation of the integral operator $T\colon \mathbb{P}_2 \longrightarrow \mathbb{P}_3$ defined by the equation $T(x^k) = \int_0^x t^k dt$ using the standard bases $\mathcal{B} = \{\mathbf{v}_1 = 1, \mathbf{v}_2 = x, \mathbf{v}_3 = x^2\}$ and $\mathcal{B}' = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x, \mathbf{w}_3 = x^2, \mathbf{w}_4 = x^3\}$, all math majors embraced a straightforward step-by-step approach similar to the integral operator one previously analyzed. Upon exploring the matrix representation both analytically and in MATLAB in a similar manner as before, math majors came up with the matrix representation A; they also felt the need to explore the integral operator and its representation further with the DGS. For that purpose, S8 entered the basis vectors of $\mathcal{B}$ as functions by respectively typing **v1(x)=1; v2(x)=x; v3(x)=x^2**; in the DGS. S8 referred to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as a constant, a line with positive slope, and a parabola, respectively. Upon the interviewer's probing how she would visualize the vector $\mathbf{v}$ whose coordinates relative to the basis is given by $[4 \quad 0 \quad -3]^T$, she entered the desired vector by typing the linear combination syntax **v=4v1-0v2-3v3**, whose graph S8 referred to as an upside-down parabola (fig.5a). She then typed the command **f=integral(v)**, resulting in the curve with purple color (fig.5b). Next, she focused on the representations. S8 introduced the matrix A of the integral operator T which she had previously obtained algebraically by typing the syntax **A={{0,0,0}, {1,0,0}, {0,1/2,0}, {0,0,1/3}}**. She then entered the coordinate representation of vector $\mathbf{v}$ relative to the standard basis $\mathcal{B} = \{1, x, x^2\}$ for $\mathbb{P}_2$ using the syntax **vB = {{4}, {0}, {-3}}**. Next, she typed **wB'= A*vB** and explained that wB' represented the coordinate representation of the sought vector relative to the standard basis $\mathcal{B}' = \{1, x, x^2, x^3\}$ for $\mathbb{P}_3$ which should coincide with the graph of the purple curve when plotted (fig.5c). In the last step of her visualization, she entered the sought vector by paying close attention to its coordinate representation by typing **g=0*1+4*x+0*x^2-1*x^3** and commenting that the two graphs should match, a result she indicated she was expecting (fig.5d).

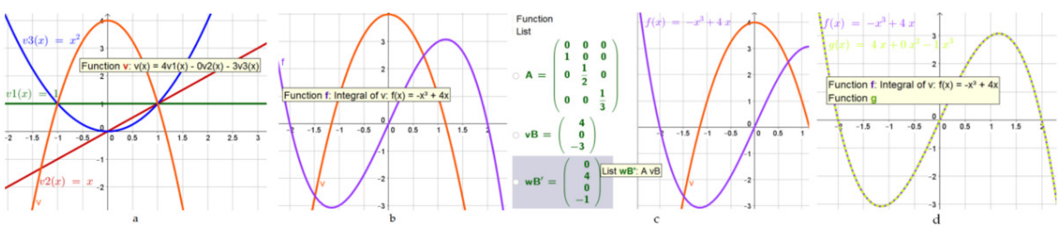

Figure 5. S8's visual approach for Task 6.

Conclusion

The findings highlighted in the present report are in agreement with Hillel's (2000) observations regarding the importance of correctly finding a matrix representation of a linear transformation relative to a basis; the report's findings are also in agreement in that math majors were also "able to think about matrix representations of linear operators as objects of inquiry in their own right" (p.199). Furthermore, math majors demonstrated representational creativity which enabled them "consider the general conditions under which a linear operator can have a particularly desirable representation" (p.199). In order to help linear algebra students understand how objects and their representations interact in analogy, a CAS such as MATLAB or DGS such as GeoGebra could be very beneficial. This has a pedagogical implication that aligns with the Linear Algebra Study Group (LACSG) recommendation for use of technology and computer software: "Faculty should be encouraged to utilize technology in the first linear algebra course" (Carlson, Johnson, Lay & Porter, 1993, p.45). Because basis-dependent representations are not unique, generating object-representation analogy maps in a DGS-MATLAB facilitated learning environment could help linear algebra learners make sense of what stays invariant.

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