Developing Pre-Service Elementary Teachers' Conceptions of Place Value Through Coursework

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This study investigates how instruction in the base-8 number system in a Calculus Course for pre-service elementary teachers (PSTs), affects PSTs' conceptual understanding of place value. Despite the importance of number sense and numerical operations in the K-12 curriculum, research shows that many PSTs struggle to grasp place value conceptually, often focusing on procedural rather than conceptual knowledge. This study uses a convergent mixed-methodology approach with quantitative pre- and post-tests and qualitative clinical interviews to analyze PSTs' understanding of place value after instruction in base-8. Findings suggest that instruction in base-8 improves the sophistication of PSTs' place value conceptions, though verbalizing that understanding remains a challenge for some. These results highlight the need for a focus on both conceptual understanding and consistent language use in undergraduate teacher preparation programs.

Keywords: pre-service elementary teachers, content knowledge, place value, undergraduate mathematics

The concept of number sense and operations is one that permeates all levels of K-12 mathematics curriculum in the United States (National Governors Association & Council of Chief State School Officers, 2010). As such, it is imperative for elementary school students to develop proficiency with number sense and numerical operations (Kilpatrick et al., 2001). Despite its clear importance, research suggests that undergraduate pre-service elementary teachers (PSTs) struggle to understand numerical operations and place value at a conceptual level (e.g. Southwell & Penglase, 2005; Thanheiser, 2009b).

Often PSTs are proficient in carrying out algorithms for numerical operations, but struggle to conceptually explain the algorithm or why it works (Ball, 1989; Ma, 1999; Thanheiser, 2009b). These struggles are sometimes left unresolved even by the end of PSTs' undergraduate teacher preparation program (Thanheiser, 2009b). Fortunately, some research suggests that studying alternate numeration systems can deepen PSTs' conceptual knowledge of place value (e.g. Yackel et al., 2007; McCain, 2013; Thanheiser et al., 2013a; Thanheiser & Melhuish, 2019). The purpose of this study is to analyze PSTs' conceptual understanding of place value after in-depth instruction on alternative numeration systems and basic operations in the base-8 number system.

Literature Review

PSTs often exhibit a calculational perception of mathematics (Thompson et al., 1994; McClain, 2003) and see the discipline as simply "memorization of basic facts, rules, and procedure[s]" (Simonsen & Teppo, 1999, p. 517). This perception can lead PSTs to engage with arithmetic operations on multi-digit numbers as surface-level manipulations of algorithms (McClain, 2003; Tirosh & Graeber, 1989). While the procedural treatment of standard algorithms may be a natural consequence of years of use, the tendency to focus on procedural aspects of algorithms over conceptual understanding persists when PSTs are introduced to novel alternative algorithms (Harkness & Thomas, 2008; Lo et al., 2008; Nugent, 2007). It is common for PSTs to conflate procedural accuracy and conceptual understanding of the underlying mathematics (Graeber, 1999). As such, while PSTs typically have no problems executing the standard arithmetic algorithms, they struggle to justify why they work (Ball, 1989; Ma, 1999; Thanheiser, 2009b). Southwell & Penglase (2005) argue that some misconceptions around arithmetic algorithms originate from a lack of conceptual understanding of place value. Upon entering undergraduate teacher preparation programs, Thanheiser (2009a) found PSTs often hold incorrect conceptions of place value, which can persist through their undergraduate education (Thanheiser & Philip, 2012; Thanheiser et al., 2013b; Thanheiser, 2009b).

Research suggests that studying alternative numeration systems can help correct these misconceptions and deepen PSTs' conceptual understanding of place value. Thanheiser & Melhuish (2019) designed an instructional sequence to highlight the three key features of the base-10 number system - grouping, having a base, and place value - by studying other numeration systems without these features (e.g., tally & Egyptian). Their findings suggest that coupling this approach with the study of arithmetic algorithms using the Mayan numeration system (a foreign system possessing the three key features previously listed), left PSTs with a more robust understanding of place value in the base-10 number system (Thanheiser & Melhuish, 2019). Other research supports the idea that the study of algorithms in alternate numeration systems can help PSTs develop their conceptual understanding of arithmetic algorithms (McCain, 2003), place value (Thanheiser, 2015), number relationships, and grouping strategies (Yackel et al., 2007).

Given the promising results that engaging PSTs in instruction of alternative numeration systems has on their conceptions of place value, this study adds to that body of work by operationalizing an established framework for categorizing PSTs' conceptions of place value (Thanheiser, 2009b) to evaluate the efficacy of in-depth instruction on the base-8 number system and use of the standard arithmetic algorithms within that system to increase the sophistication of PSTs' conceptions of place value. In this instructional unit, PSTs learn how to count, represent numbers, add, subtract, and multiply in base-8 to experience and illuminate elementary students' struggles with place value when learning arithmetic algorithms for the first time. The features of the base-8 number system are identical to that of base-10 with the exception of the base, much closer than numeration systems such as the Mayan numeration system, providing familiarity to PSTs while also forcing attention to be on place value as this is the sole featural difference between the two systems. Specifically, this study aims to answer the following research questions:

1. Does studying arithmetic in base-8 increase the sophistication of PSTs' conceptions of place value with statistical significance?
2. To what extent do the qualitative findings converge or diverge with quantitative findings?

Conceptual Framework

For this study, we apply Thanheiser's (2009b) framework for the categorization of place value conceptions. Thanheiser's framework (2009b) was developed from interviews conducted with 15 undergraduate PSTs before their first undergraduate mathematics course. Participants engaged in two clinical interviews over four weeks. During these interviews, they were given problems to solve. This led to Thanheiser (2009b) developing four groups to categorize PSTs' conceptions of place value. The framework was created to "build on aspects of the framework for two-digit numbers (Fuson et al., 1997) and extend them to numbers of three or more digits on the basis of the results of [the] study" (Thanheiser, 2009b, p. 261). The framework is summarized in Table 1. It is important to note that the conceptions are listed in order from least sophisticated to most sophisticated and that the first two are incorrect conceptions, while the final two are correct conceptions (Thanheiser, 2009b).

Table 1. Explanations of Place Value Conceptions held by PSTs

Conception Held
Concatenated digits only. PSTs holding this conception conceive of all the digits in terms of only ones (e.g., 548 would be 5 ones, 4 ones, and 8 ones).
Concatenated digits plus. PSTs with this conception conceive of at least one of the digits in the number in terms of an incorrect unit type at least some of the time. They therefore struggle when relating the values of the digits in a number to one another. A PST may correctly conceive of groups of 100 ones for a digit in the hundreds place but incorrectly conceive of one for the tens place (e.g., 389 would be seen as 300 ones, 8 ones, and 9 ones).
Groups of ones. PSTs with this conception reliably conceive of all digits in terms of groups of ones; 389 would be 300 ones, 80 ones, and 9 ones. PSTs holding the conception do not conceive of the digits in terms of reference units.
Reference units. PSTs with this conception reliably* conceive of the reference units for each digit in the number and can relate the reference units to one another; in 389, the 3 can be seen as 3 hundreds or 30 tens or 300 ones, and the 8 can be seen as 8 tens or 80 ones.
**Reliably* in these definitions means that after the PSTs were first able to draw on a conception in their explanations in a context, they continued to do so in that context.

Note. Adapted from Thanheiser (2009b, p. 263).

Methods

This study followed a convergent mixed-methodological design (Creswell & Clark, 2017). Quantitative secondary data was collected from students in the form of a base-10 pre- and post-test completed as part of the work required for their undergraduate mathematics course. Additionally, clinical interviews were used for qualitative analysis to further elicit student thinking and validate results (Creswell & Clark, 2017).

Participants

This study was conducted on elementary PSTs (n=37) at a large southeastern university in the spring semester of 2024. Participants were second-semester freshmen enrolled in a calculus course specifically designed for PSTs. The calculus curriculum for this course includes a unit on alternate numeration systems and arithmetic in base-8. These systems included the tally system, the Mayan system, the Egyptian system, and more. Students spent the next two weeks studying arithmetic in base-8. In particular, students relearned how to count, add, subtract, and multiply in base-8. The first author served as the instructor for the mathematics course. Throughout instruction, emphasis was placed on the concept of place value and what each digit represents.

Data Collection

Students were given a paper-pencil pre-test prior to any instruction. Immediately following the instructional sequence described above, students were given the same assessment as a post-test. The assessment was adapted from the instrument used by Thanheiser (2010) to classify elementary PSTs' conceptions of place value by level of sophistication. As such, its intent was to measure the participants' demonstrated level of sophistication of place value conception. The pre- and post-tests of students who consented to participate (n=43) were used for analysis and any participants missing either a pre- or post-test (n=6) were excluded.

A subset of five students participated in a clinical interview (Hunting, 1997) to explore their conceptions of place value. However, two of the students interviewed were participants missing pre- and post-test data that were excluded, resulting in three interviews for analysis. Interviews were conducted by the second author at the end of the spring semester, approximately three months after instruction. During the interview, participants were asked to verbally explain their thinking as they solved one addition, one subtraction, one multiplication, and one division problem in base-10. Interviews were semi-structured (Maher & Sigley, 2014) and students were asked probing questions throughout the interview to elicit their thinking (Hunting, 1997).

Each interview lasted about 30 minutes and was conducted in person. Interviews were recorded via Zoom and the students' work was completed on an iPad with its screen shared to the Zoom meeting. In this way, both the participants and their work were recorded simultaneously. Transcripts of the interviews were created with screenshots of participant work for data analysis.

Data Analysis

Analysis occurred at the end of the semester after all data collection was complete, in line with a convergent mixed methodology design (Creswell & Clark, 2017). This allowed participants to remain unknown to the instructor until the conclusion of the course. Thus, the quantitative and qualitative data were analyzed concurrently as separate data sets.

Quantitative analysis. A scoring system that built upon the levels of Thanheisers' (2009b) framework (0 - *irrelevant,* 1 - *concatenated digits,* 2 - *concatenated digits plus,* 3 - *groups of ones,* 4 - *attempt at reference units,* 5 - *reference units*) was used to assign a score to each question on each pre- and post-test. Authors scored all pre- and post-tests individually with a high percent agreement (79.72%) and met to resolve disagreements. A final average pre-and post-test score was calculated for each student and used for statistical analysis. SAS Studio was used to run paired *t*-tests to determine if there were statistically significant differences between student pre- and post-test scores.

Qualitative analysis. Each interview (n=3) was transcribed and coded for instances of each type of place value conception. Since the primary goal of the study was to analyze PSTs' conceptions of place value, moments where participants were discussing place value were identified within the transcripts and assigned a conception type using Thanheiser's (2009b) levels of sophistication as an a priori framework (Miles et al., 2018). It is worth noting that Thanheiser's levels are intended to describe how PSTs *reliably* conceive of place value; however, throughout coding, the levels were used to describe *individual* instances of place value conceptions. Authors coded independently and met to establish agreement.

Data integration. After initial analysis of quantitative and qualitative data was complete, qualitative data was transformed to quantitative data (Creswell & Clark, 2017) by counting the frequency of instances in which participants displayed each type of conception. Frequencies were then used to calculate percentages to identify what portion of the time participants used each conception. For the participants with both interview and pre-post data (n=3), the post-test scores were compared with the frequency of each conception they displayed in clinical interviews. This comparison was conducted in line with convergent mixed-methodology design principles (Creswell & Clark, 2017) to corroborate results and provide more insight into participants' conceptions of place value.

Findings

This section presents the findings from analyzing the pre- and post-tests and the interview transcripts. It is important to note that, while reliably demonstrating the same kind of conception is a cornerstone of Thanheiser's (2009b) conception categorization, in this study, every instance of place value conception used by each participant has been counted and categorized.

Quantitative Findings

To answer research question 1, participant pre-test averages (μ=2.00, σ=0.94) on the base-10 assessment were compared with post-test averages (μ=2.89, σ=0.91) in a paired t-test. Results indicate that the mean level of sophistication displayed for participants on the post-test was statistically significantly higher compared to the pre-test and had a large effect size (t=7.00, p<0.0001, d=1.15). This suggests that PSTs' level of sophistication of place value conceptions improved after instruction on arithmetic in base-8.

Qualitative Findings

The percentages of frequencies with which each interviewed participant demonstrated each conception in their interviews are displayed in Table 2 along with their base-10 post-test average and gain score. Note that the Concatenated Digits Plus code was only applied for instances where participants used both Concatenated Digits and a correct conception (either Groups-of-Ones or Reference Units) within the same sentence. As shown in the table, results of the clinical interview analysis are in line with the results of the post-test for both Student 1 and 2. Student 3's interview, on the other hand, was not necessarily consistent with their post-test score.

Table 2. Integrated Qualitative and Quantitative Results

Percentages of Each Conception, Post-Test Scores, & Gain						
Participant	Concatenated Digits	Concatenated Digits Plus	Groups-of-Ones	Reference Units	Post-Test Scores	Gain Scores
Student 1	16.13	6.45	29.03	48.39	4.6	1.6
Student 2	63.16	0.00	15.79	21.05	2.8	1.2
Student 3	46.43	7.14	28.57	17.86	4	0.6
Total	38.46	5.13	25.64	30.77		

Student 1 scored high on the post-test (μ=4.6 out of 5), demonstrated a correct conception of place value over three-quarters of the time in her clinical interview (77.42% of all instances), and showed the most sophisticated conception (Reference Units) almost half of the time (48.39%). For example, Student 1 used Reference Units language when she tackled the task of subtracting 40 from 10 (see the left-most picture in Figure 1), to describe extracting a group of one hundred from the hundreds place, augmenting the tens place to be 110 minus 40, stating, "So I take one of the 100 groups, making that 500, and then adding a one in front of the tens, since it would be 10 groups of 10." Student 1 is able to refer to numbers by their groups and change that language to reflect the place value she was working with, a group of 100 becomes 10 groups of 10. After regrouping, when Student 1 performed the subtraction in the same figure, she switched to using a Groups-of-Ones conception. When subtracting 40 from 10, Student 1 explained her reasoning as, "Technically, it's a hundred…and then I'm putting it over here, which would make it…110 minus 40 instead of 20 minus 40." This differs from referring to the numbers as four tens or 11 groups of ten but is still correct.

The results are similarly aligned for Student 2 who scored relatively low on the base-10 post-test (μ=2.8 out of 5) and demonstrated incorrect conceptions of place value over half of the time in her clinical interview (63.16%). For example, when Student 2 subtracted 40 from 10, she used a Concatenated Digits level of conception. Explaining her steps in the middle picture of Figure 1, Student 2 said, "Then we are going to use this 10 and subtract 4, and that would be 6. Then I do 6 minus 2, just like earlier, to get 4," explaining the steps as if she were dealing with the one's place rather than the ten's or hundred's place.

For Student 3, the quantitative and qualitative results were misaligned. The post-test score was high for Student 3 (μ=4 out of 5). However, she demonstrated correct conceptions of place value less than half of the time (46.43%) and her most frequently demonstrated conception was Concatenated Digits (46.43%). To describe multiplying 700 with three in the third picture of Figure 1, Student 3 explained, "So then, when we multiply 7 times 3, you're getting what was essentially 2,100." The beginning of this statement, referring to each number as seven and three shows the use of a Concatenated Digits level of thinking, yet then recognizing that the multiplication actually yields 2,100 employs a Groups-of-Ones level of thinking. This is one example of an instance coded as a Concatenated Digits Plus conception. The contradictory use of language will be examined further in the discussion section.

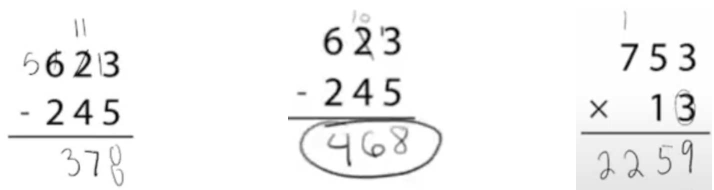

Figure 1. Student Examples of Conceptions: Student 1 on Left Using both Reference Units and Groups-of-Ones, Student 2 in Middle Using Concatenated Digits, Student 3 on Right Using Concatenated Digits Plus

Discussion

The findings suggest that the study of arithmetic algorithms in base-8 has a statistically significant impact on PSTs' level of sophistication of place value conception. However, with an average gain score of 0.89 points, the small amount of gain suggests that engaging in arithmetic in base-8 may not be sufficient to drastically change PSTs' conceptions of place value. While the pre- and post-tests offer insight into the change in the thought process of students before and after engaging in arithmetic algorithms in base-8, they also provide only a snapshot of student thinking. The interviews with a select number of participants gave greater insight into students' understanding of place value as they approached arithmetic operations. The close alignment of qualitative and quantitative findings from Students 1 and 2 provides further support for this improvement.

The misalignment of quantitative and qualitative data from Student 3 highlights an interesting additional consideration. During her interview, Student 3 switched between correct and incorrect language within single statements in a way that could be more confusing than consistently using a single conception - even an incorrect one. For example, when multiplying 700 and 3, she explained "when we multiply 7 times 3, you're getting what was essentially

2,100.” For a student who has just learned 7 times 3 is 21, not 2,100, this can be more difficult than either using Groups-of-Ones language - “700 times 3, which is 2,100” - or Concatenated Digits language - “7 times 3, which is 21.” It is clear from the post-test scores that Student 3 conceptually understands place value; however, she demonstrated difficulty with the consistency of language use while explaining problems in real time. This highlights the importance of having PSTs practice verbally explaining arithmetic operations using correct language. Though studying arithmetic in an alternative numeration system may deepen PSTs’ understanding of place value and arithmetic algorithms, that does not guarantee they are able to verbalize that understanding to their future students.

An interesting finding that emerged from the interview data was students’ response to re-learning arithmetic and place value in base-8. For example, Student 2 referenced that the instructional unit had changed her understanding of place value and that previously, for “all problems in base-10, [she’d] learned to just accept it as it is” and that carrying out algorithms in base-8 had “tricked her brain” in a way. Similarly, Student 3 reflected that “I did not have that basic understanding of place value going into this class… you know I would have never told you that when you multiply this 3 and this 1 [referring to a double-digit multiplication problem], you’re multiplying 3 times 10”. Additionally, Student 1 indicated that after “relearning addition, subtraction, [and] multiplication… it made me kind of understand why that stuff works more so”. Student 1 also explained how she was able to easily transfer her learning in base-8 to base-10: “When I was having trouble, thinking… as soon as you hit eight, that’s actually one-zero because it’s one group of 8. I was like, okay, well, in tens when you hit 10, that’s one group of [the base]”. It is possible that the nature of base-8 and its similarity to the base-10 system made it easier for the PSTs to develop a conceptual understanding of place value and to disrupt the automatic procedural nature of their thoughts surrounding operation algorithms within base-10.

Limitations

This study was limited by a sample size that, while large enough to produce statistically meaningful quantitative results, was still small. Additionally, the involvement of the first author instructing the course naturally introduces the chance for bias in the results. The post-test was given immediately after the participants had engaged in the base-8 unit and interviews were conducted within the same semester. While the base-8 unit clearly changed the way participants described arithmetic operations, whether these changes are sustained remains to be seen. Finally, whether base-8 instruction is more beneficial than less familiar systems, such as the Mayan or Egyptian system, would require further study.

Conclusion

This study highlights the positive impact of studying arithmetic in the base-8 system on PSTs' conceptual understanding of place value. While the instructional unit successfully improved participants’ level of sophistication of place value conception, the variation in individual results underscores the complexity of this learning process. Some participants exhibited strong conceptual understanding, yet struggled to consistently verbalize their thinking, emphasizing the need for more focus on clear communication of mathematical concepts. Future research should further investigate the long-term effects of learning alternative numeration systems on both conceptual comprehension and the ability to articulate mathematical reasoning.

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