Calculus I Students' Logical Reasoning About the Definition of Continuity of Function

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We examine undergraduate calculus students' reasoning about the continuity of a function at a point. The study was conducted in the context of an NSF-funded project that supported the redesign of Calculus I recitations at a large public university. One aspect of the reform effort involved introducing conceptually rich, collaborative activities into Calculus recitations. One such activity focused on the concept of continuity at a point, with one particular task probing students' understanding of the definition of continuity at a point using true-false statements. Data from 84 student in-class worksheets were analyzed with a focus on the type of reasoning students used to respond to these questions. The results reveal a variety of graphical, logical, example-based, and rules-based strategies.

Keywords: Continuity at a point, Limits, Calculus learning, Logical reasoning

The concepts of limits and continuity are fundamental to calculus, yet they present many challenges for students to understand practically and conceptually (e.g., Amatangelo, 2013; Bezuidenhout, 2001; Denbel, 2014; Williams, 1991). Some researchers locate these challenges in the procedures-oriented emphases that are prevalent in many calculus textbooks (Bezuidenhout, 2001), while others suggest that students' challenges may reflect the historical, mathematical, and psychological complexity inherent in these concepts (Juter, 2006; Tall & Katz, 2014). Consequently, many studies focused on revealing various types of students' conceptions and misconceptions about limits and continuity. Our study seeks to add to this literature. In particular, we examine how first-year Calculus students reason about the definition of continuity at a point as they learn about this concept in an authentic classroom setting.

The study was conducted in the context of an NSF-funded project aimed at improving the teaching of Calculus. The enacted changes included using more conceptually oriented tasks, which students solved by working in small groups during the recitations. This context constitutes an authentic learning setting where students, after attending a lecture on continuity, work together to solidify their understanding of this concept. Learning construction is expected to be characterized by students expressing incorrect or incomplete ideas, drawing on prior knowledge and naïve conceptions that may or may not apply to the mathematical situation at hand. Within this learning situation, we explore the research question: *How do students reason about the definition of continuity at a point, and what types of justifications do they use to make sense of this definition*?

Literature Review and Theoretical Background

Definition of Continuity at a Point

The formal definition of continuity at a point states that a function f is continuous at a point c, in its domain, if for every $\varepsilon > 0$ there exist a $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. This definition is rarely introduced to first-year Calculus students. Instead, a more intuitive definition is commonly used: a function f is continuous at a point c if $\lim_{x \to c} f(x) = f(c)$. While this definition relies on notation and concepts that are more familiar to students, it has some non-obvious challenges. First, it is often written as a conditional rather than a bi-conditional statement, which may affect how students interpret and apply the definition in practice. Second, $\lim_{x \to c} f(x) = f(c)$ implies that a function is defined at a point c, and the limit $\lim_{x \to c} f(x)$ exists. Thus, some textbooks, introduce a *criterion* for determining if a function is continuous at a point *c*, comprised of three rules: (1) $f(c)$ is defined, (2) $\lim_{x \to c} f(x)$ exists, and (3) $\lim_{x \to c} f(x) = f(c)$. This three-rule structure streamlines the procedure for determining if a given function is continuous at a point, but complicates the logical structure of the definition. Let V (value), L (limit), and E (equal) represent propositions corresponding to the three rules above, and C (continuity) the conclusion. Then the definition of continuity at a point has a logical form of $(V \wedge L \wedge E) \Leftrightarrow C$. The criterion for continuity at a point is the direction $(V \wedge L \wedge E) \Rightarrow C$, while the contrapositive $(\neg C \Rightarrow (\neg V) \vee (\neg L) \vee (\neg E))$ represents the property of a function being not continuous at a point. While first-year Calculus students are not expected to unpack this complex logical structure, they may be expected to use it flexibly and implicitly when solving problems. For example, students must understand that a function can be discontinuous at a point, even if it is both defined at that point and has a finite limit at that point.

Students' Reasoning about Continuity at a Point

The concepts of limit and continuity are closely related to each other and are the core of differential Calculus. However, they pose persistent difficulties to learners (Núñez, et al., 1999; Oehrtman, 2009; Tall & Vinner, 1981). Studies distinguish between how students evaluate limits and continuity at a point (often utilizing algorithmic, procedural approach) vs. how students interpret them conceptually (Amatangelo, 2013; Bezuidenhout, 2001; Denbel, 2014).

When reasoning about limits and continuity, students rarely rely on the formal definition, even if they are familiar with it, but rather on what Tall and Vinner (1981) call a personal concept image – a collection of all mental images, words, metaphors, and associations a student has developed about a particular mathematical concept. Thus, students may interpret continuity using intuitive, dynamic images of graphs that can be drawn without lifting a pen or interpret limits using the language of movement, e.g., "getting close." Such reasoning is associated with a dynamic conception of limits and continuity and can lead to either correct or incorrect conclusions (Núñez, et al., 1999; Oehrtman, 2009; Przenioslo, 2004). For example, Denbel (2014), found that pre-engineering students often viewed limits as "unreachable" or "approximations", and many mistakenly believed that if the limit exists at a point, then the function is continuous at that point. Similar misconceptions were observed among first-year Calculus students in South Africa (Bezuidenhout, 2001) and Sweden (Juter, 2006).

Amatangelo (2013) identified four common misconceptions related to limits and continuity at a point: (1) a function defined at a point implies continuity at that point, (2) the limit value is equivalent to the function value at a point, (3) the existence of a limit implies continuity, and (4) a limit at a point necessitates discontinuity. That study involved over 800 students and revealed that these misconceptions were prevalent and persistent until students took advanced calculus courses like Multivariate Calculus or Theory of Analysis.

Logical Reasoning

As mentioned above, interpreting the definition of continuity at a point requires (possibly implicit) reasoning with logical operators of conjunction ("and"), disjunctions ("or"), and negation. Most first-year calculus students have not received explicit training in mathematical logic. Thus, their conceptions of logical principles are rooted in everyday language that often misaligns with conventional mathematics (Epp, 2003). For example, students tend to interpret disjunctions as exclusive rather than inclusive "or" (Dawkins & Cook, 2017) and confuse between the statement and its converse (Durand-Guerrier, 2003). Additionally, negations and reasoning with contrapositives have been shown to be challenging for students (Buchbinder & McCrone, 2019; Brown, 2018; Yopp, 2020). While the logical form of a statement may affect how it is interpreted, some psychologists argue that people mainly reason by analyzing the *semantic* content of the statements: by forming and mentally manipulating mental representations of that content (Johnson-Laird & Byrne, 2002; Oaksford & Chater, 2002). Bringing back the notions of concept image and dynamic reasoning, this may imply that when interpreting the definition of continuity at a point, students form mental images of continuous and discontinuous functions and mentally manipulate them to arrive at certain conclusions.

Method

This study is a part of the larger, NSF-funded project that focused on reforming the teaching and learning of STEM gateway courses, particularly courses in the Calculus sequence, at a large public university in the northeast of the USA. The reform efforts involved re-envisioning of Calculus I recitations by introducing conceptually rich activities, active-learning strategies such as collaborative problem-solving (CBMS, 2016), and introducing learning assistants (LAs) alongside graduate teaching assistants (GTAs) to support student group work. Calculus I is taught in the lecture-recitation format. Large lectures (~120 students) are taught by a faculty member, and coordinated recitation sections of about 20 students each are led by GTAs.

Data Collection

The data for this study were collected in Spring 2023 during one of the first recitations of the semester. The topic of the activity was the continuity of functions. We focused on the task, modified from Straumanis et al. (2013), that probed students' understanding of the definition of continuity at a point by asking students to respond to five true-false statements (Figure 1).

Given five statements. For each statement, decide if it is true or false. For each false statement, provide a counterexample: sketch a graph of a function showing that the statement is false. Then, use another representation: a table, a symbolic equation, and/or words to explain why the graph you sketched disproves the statement.

1. If $\lim_{x \to c} f(x) = f(c)$ then f is continuous at $x = c$.
2. If f is not continuous at $x = c$, then $f(c) \neq \lim_{x \to c} f(x)$.
3. If f is not continuous at $x = c$, then $f(c)$ is undefined.
4. If f is not continuous at $x = c$, then $\lim_{x \to c} f(x)$ does not exist.
5. If f is not continuous at $x = c$, then either $f(c)$ is undefined, or $\lim_{x \to c} f(x)$ does not exist.

Figure 1. Definition of Continuity at a Point Task

Item (1) is a true statement, which is an abbreviated version of a semiformal definition of continuity at a point. The hypothesis $\lim_{x \to c} f(x) = f(c)$ suggests that both the limit at point *c* and the value of the function at point *c* exist and are equal to each other, and therefore, the function is continuous at *c*. Item (2) is the contrapositive of item (1) and is logically equivalent to it, thus it is also true. Items (3-5) contain false statements comprised of the elements of the contrapositive; the conclusion of item (5) is a combination of items (3) and (4).

Data Analysis

To analyze the data—students' written responses to the task—we used open coding (Patton, 2002), informed by the literature described in the theoretical background section. While we noted whether students responded correctly or not to each question, our main focus was on the reasoning they used to justify their responses. The following questions guided our analysis: Did the student's explanation rely on graphs, words, or both? If graphs were used, what kinds of graphs were they? If discontinuities were mentioned (verbally or graphically), what kinds of discontinuities were they? Were their reasoning strategies rule-based, or did they reason about the semantic meaning of the statements (or both)? Were students' justifications logically sound (did they reach valid conclusions based on true premises)? If logical mistakes were made, what was the nature of these mistakes?

Students' work for each item (Figure 1) was tagged and grouped in different ways according to the answers to these questions. Next, we examined trends in data across all five sub-questions. Due to space constraints, in this paper, we report mainly on the phenomena identified in student responses to individual items, followed by a discussion of general outcomes about students' reasoning about the definition of function's continuity at a point. Before describing the results, we emphasize that our goal in this analysis was not to make any assertions about students' conceptions or cognitive processes. Rather, we seek to understand how students reason about the definition of continuity at a point and how they justify their thinking.

Results

Overall, students were relatively successful in responding to all items in Figure 1.

Item 1, which presents a definition of continuity at a point had a 100% success rate. Four out of 84 students initially selected "false" but crossed it and wrote "true," possibly after discussing it with their peers. Most students' justifications (52) mentioned the definition of continuity at a point in some way. Of those, 17 students re-formulated the definition of continuity in their own words (e.g., "the function is continuous because the limit and the function value agree at a point"). 11 students wrote that the statement is true based on the "three rules of continuity" and stated these rules. We interpret these responses as rules-based reasoning. 19 students drew graphs of continuous functions in their responses, but only one student explained how their graph relates to the statement at hand. It seems that in these cases, the graphs served as illustrations, and students may not rely on these graphs to reason about the meaning of the statement, especially if they recognized it as a definition of continuity at a point.

11 students mentioned that given the statement, there cannot be any discontinuity at $x = c$ and therefore, the function was continuous at $x = c$. This is an attempt to apply contrapositive reasoning, where the negation of the conclusion implies the negation of the condition, or "eliminating counterexamples" (Yopp, 2020). While this reasoning could be valid, most students did not specify how they knew that there were no discontinuities, but some mentioned the different types of discontinuities they learned (hole/removable, jump, infinite).

Six students exhibited a "defined = continuous" misconception, previously identified in the literature (e.g., Amatangelo, 2013). For example, one student wrote: "If we are able to plug c into the function $f(x)$ then it must be continuous because it can approach that limit."

One student claimed that statement 1 is true unless the function and the limit are both undefined and therefore "equal DNE." The student seems to treat the acronym DNE (does not exist) as a number, which can be equal to another DNE "number," resulting in an equality

$\lim_{x\to c} f(x) = DNE = f(c)$. The student realized that a function cannot be continuous in such a situation but thought it implies that equality is true and should be treated as an exception.

Although this was a singular occurrence, it indicates the ambiguity of notation and a potential misinterpretation of what DNE stands for (undefined for a function and does not exist for a limit) and that the acronyms cannot be treated or equated as numbers.

Item 2 is a contrapositive of item 1. Despite the high success rate (80 students correctly responded "true"), the analysis suggests that it was a much more difficult item to comprehend compared to item 1. Of the students who correctly determined the statement was true, 11 merely restated it as their explanation. 23 students mentioned at least one rule of continuity. Of those, seven claimed that "if a function is discontinuous at a point c, it means it is undefined at that point," and three students said that "it means that $\lim_{x\to c} f(x)$ does not exist." Possibly, the students were showing different ways in which a function can be discontinuous, but it can also indicate a limited concept image of discontinuity, with students overlooking other ways in which a function can be discontinuous at a point. For example, one student wrote: "It is not continuous at $f(c)$ then that means there's a jump or stop somewhere, meaning the limit can't exist."

31 students drew at least one discontinuous graph in their explanation. Of those students, 24 drew a graph of a function with a hole or a removable discontinuity, while some students drew multiple graphs with different types of discontinuities. While these were correct discontinuous graphs, it is unclear whether students considered them as illustrations of where the statement is true or if they attempted to "prove" the statement with specific examples of discontinuous functions, thus exhibiting example-based reasoning (Rissland, 1999; Watson & Mason, 2006).

Rather than examining discontinuities, 21 out of 84 students used a rules-based approach. Working from the three-rule criterion of continuity, these students explained that since the function was discontinuous at $x = c$, it must not satisfy one of the three rules of continuity at that point. While this is a potentially productive approach, some students interpreted it narrowly, identifying specific rules that would be broken.

When students mentioned all three rules of continuity at a point, they seemed to struggle with negating their conjunction. For example, one student wrote: "Because the rule of continuous function does not allow $\lim_{x\to c} f(x) \neq f(c)$ to be continuous, all rules can be proven to be untrue." The student could be wrongly assuming that all three continuity rules must be simultaneously false or merely using "all" in the sense of "any" — a common confusion (Epp, 1999).

Six students justified statement 2 by proving its converse: If $f(c) \neq \lim_{x\to c} f(x)$, then the function is not continuous. Since definitions are true, biconditional statements, the students reached the right conclusion, even by using a logically problematic approach.

Items 3 and 4 are both false. Students performed well on both, using similar reasoning and exhibiting similar challenges. Most students (57 for item 3, and 53 for item 4) drew graphs showing counterexamples to justify their reasoning. Of these students, most (37 of 57 for item 3 and 38 of 53 for item 4) drew a graph of a function with a removable discontinuity at $x = c$. For item 3, these students drew graphs of functions where $f(c)$ was defined, and for item 4 $f(c)$ was not defined. Other students (33 for item 3, and 29 for item 4) mentioned different types of discontinuities which would constitute a correct counterexample to this statement. Similarly with the graphs, most of these students (24 for item 3, and 29 for item 4) mentioned removable discontinuities to prove this statement false. Overall, very few of the common misconceptions suggested in the literature appeared in these explanations.

Item 5 was the one the students found most challenging compared to other statements. 71 out of 84 students correctly responded “false,” with 14 of them initially choosing the option “true”. Many students noticed that item 5 is a combination of items 3 and 4 and used that in their justifications. They did this in a variety of ways, not always correctly. Most students either explicitly referred to or re-drew the graphs they used to disprove items 3 and 4: a graph of a discontinuous function with a hole (showing $f(c)$ undefined) or jump discontinuity with $f(c)$ defined but undefined or non-existing limit. The challenge appeared in making conclusions based on these examples. 11 out of 84 students incorrectly inferred from these graphs that the statement was true, not realizing they were “proving” the statement with supportive examples.

One possible reason for these types of responses could be students’ limited concept image of discontinuity, restricted to only certain types, while a removable discontinuity would constitute a valid counterexample. Indeed, the majority of correct arguments relied on graphs with removable discontinuity, like Figure 2, where the student explained that “The graphed function is not continuous because $f(c) \neq \lim_{x \to c} f\,(x)$, but both $f(c)$ and $\lim_{x \to c} f\,(x)$ exists.”

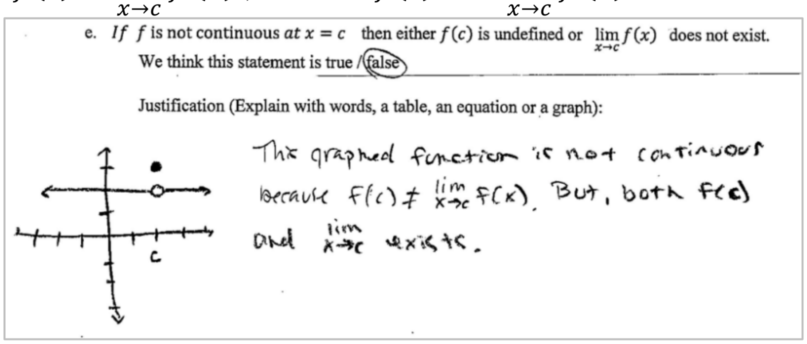

Figure 2. Correct response to item 5: a counterexample of removable discontinuity

Another explanation for incorrect conclusions about discontinuous functions whose graphs contain holes or jump discontinuities could be that the students were attempting to evaluate the converse of statement 5. That is, looking at graphs where the conclusion is false and inferring the function is not continuous. Whether this is the case or not was not always clear from written responses, but the argument in Figure 3a supports the assumption about evaluating the converse.

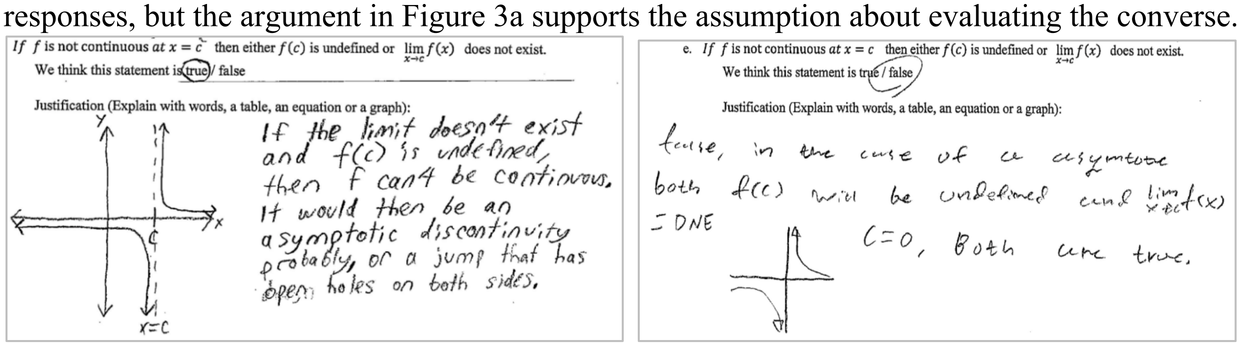

Figure 3. Different incorrect inferences from an asymptotic discontinuity. 3a (right): Evaluating the converse to “prove” the statement. 3b (left): Relying on exclusive disjunction to “disprove” the statement.

The student gave an example where both parts of the conclusion are true: “The limit doesn’t exist, and $f(c)$ is undefined” but inferred that the hypothesis must be true also: “then f can’t be continuous.” Note that in the argument in Figure 3a, the student used “and” instead of “or.” This was another recurring issue of students’ difficulties with disjunction. Figure 3b shows one of the four instances of arguments where students attempted to disprove the statement by drawing a graph of a function with an asymptotic discontinuity at $x = c$ and explaining that the statement is false because “Both $f(c)$ is undefined and $\lim_{x \to c} f\,(x) = DNE$.” These students interpreted the disjunction as inclusive rather than exclusive and asserted that they found a counterexample where both parts of the statement’s conclusion were true. Whereas in fact, such an example supports the statement. Challenges with negating disjunction, interpreting its meaning, or confusing it with conjunction led to 10 instances of incorrect reasoning on item 5.

Another example of this challenge appeared when students did not rely on graphs but on three continuity rules. This reasoning involved students claiming that the statement is false because: “All three rules need to be untrue in order for a function to be not continuous. This [statement] only specifies two of the three [rules]”. Similar to the examples above, this argument shows a rigid application of continuity rules, but contrary to those examples, this is a clear invalid negation of the disjunction. It is also interesting to note that only 10 students mentioned $\lim_{x \to c} f\,(x) \neq f(c)$ in their answer to item 5 in some way, whereas in the previous items, students relied more heavily on the three rules of continuity.

Discussion and Scientific Significance of the Study

The results reported herein shed light on the variety of ways in which students reason about the definition of continuity at a point – an important concept of introductory Calculus. The results reveal that item 1, which presents a definition of continuity at a point, had a 100% success rate, while item 5—an incomplete contrapositive—had the lowest success rate. Some of the students’ responses align with the common misconceptions identified in the literature, like conflating $f(c)$ defined or $\lim_{x \to c} f\,(x)$ with continuity (Amatangelo, 2013; Denbel; 2014; Juter, 2006). However, most of the reasoning challenges in our study were related to logical reasoning, especially the negation of the intersection of the three continuity rules ($\neg(V \wedge L \wedge E) \equiv (\neg V) \vee (\neg L) \vee (\neg E)$), interpreting the contrapositive and disjunctions (Brown, 2018; Dawkins & Cook, 2017; Yopp, 2020). Students’ responses suggest that as the logical structure of the statement became more complex, it became more difficult for students to apply continuity rules in a rote way. It forced students to attend to the logical and semantic meaning of the statements, challenging their emergent understanding of the definition of continuity. Thus, students relied heavily on examples: graphs of discontinuous functions but often struggled to make inferences from these examples. Having limited example space (Watson & Mason, 2005) or a limited concept image of discontinuities (Tall & Vinner, 1981) often negatively affected students’ reasoning. However, even when correct examples were used, drawing correct logical inferences from them was often challenging. This concurs with prior studies showing similar challenges in other mathematical contexts, like concluding the statement is true based on several examples or proving (or disproving) the converse (e.g., Buchbinder & Zaslavsky, 2019; Rissland, 1999).

The authentic context of the study, where students work in groups trying to make sense of the definition, distinguishes our study from previously reported in the literature, making for a unique contribution and capturing student thinking during concept construction. Focusing on written responses allowed us to analyze a rather large data set, but since our analysis is limited to written responses, we could not probe student thinking further. This seems critical since some of the students’ responses were rather surprising. We intend to explore them more closely in the future.

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