Navigating the Transition from University to School: A Case Study of Incorporating Reasoning and Proving in Secondary Mathematics Teaching

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University teacher preparation programs extend significant efforts to support the development of secondary teachers' competence in teaching mathematics. However, it remains an open question whether or to what extent beginning mathematics teachers enact ambitious practices in their teaching – specifically, engaging students with reasoning and proving. We report on one teacher, Nancy, who, after two years of autonomous teaching seized the opportunity to design and enact a "proof unit," by utilizing materials from her undergraduate studies. Using the framework of practical rationality and four professional obligations, we examine Nancy's discourse over various time points on her professional journey, focusing on how she held on to the ideas from her university training, culminating in the reasoning-and-proof unit. Implications for strengthening mathematics teacher preparation beyond university programs are discussed.

Keywords: Reasoning and proof, Internship, Beginning secondary teachers

University teacher preparation programs are pivotal in strengthening knowledge and shaping the pedagogical practices of future mathematics teachers (AMTE, 2017; Wasserman et al., 2023), including their competence to integrate reasoning and proving into their teaching (e.g., Conner et al., 2014; Buchbinder & McCrone, 2020). However, transitioning from university to classroom teaching poses significant challenges for beginning teachers, who must balance conflicting commitments to both the university and their mentor teacher during student teaching (Bieda et al., 2015; Thompson et al., 2013). Stylianides et al. (2013) describe specific challenges beginning teachers face when attempting to enact reasoning and proving activities in classrooms, like teachers' limited knowledge of students' proof conceptions and pre-existing school cultures. When entering their first teaching job, teachers may be overwhelmed by the increased responsibilities and competing dilemmas (Gainsburg et al., 2012), which they need to reconcile to support students' engagement with proof (Conner et al., 2014). Stylianides et al. (2017) alert to the limited number of studies that explore the development of proof-related practices of beginning teachers longitudinally. This study attempts to contribute to this research gap.

We use a case study of Nancy (a pseudonym) to investigate beginning mathematics teachers' proof practices. As part of a larger NSF-funded study, we followed Nancy for four years, from her participation in the undergraduate capstone course *Mathematical Reasoning and Proving for Secondary Teachers (*Buchbinder & McCrone, 2023), her year-long supervised internship, and first two years of autonomous practice. In her second year, Nancy seized the opportunity to design and implement a unit centered on reasoning and proving, drawing on ideas and materials from the capstone course and the professional learning community on reasoning and proving comprised of her former classmates. Nancy's case provides a unique window into the long-term effects of university training on teaching practices. We utilize Herbst and Chazan's (2003, 2011) framework of four professional obligations (described below) to explore the following research question: *How did Nancy navigate the university-to-school transition, balance various professional obligations, and apply reasoning and proving practices in her classroom?*

Theoretical Perspectives

Reasoning and Proving in Secondary Schools

The term *reasoning and proving* denotes a collection of disciplinary practices involved in creating and validating mathematical knowledge, like identifying patterns, conjecturing, explaining, justifying, and proving (Ellis et al., 2012; Reid & Knipping, 2010). We adopt a definition of proof as "a mathematical argument for or against a mathematical claim that is both mathematically sound and conceptually accessible to the members of the local community where the argument is offered" (Stylianides et al., 2017, p. 238). That is, both formal and informal mathematically valid arguments can be accepted as proof by a particular community.

While the main purpose of reasoning and proving in classrooms is to support student mathematical learning and understanding, research consistently shows that teachers find it challenging to establish such classroom environments (Stylianides et al., 2017). One of the challenges may be rooted in teachers' limited knowledge of proof or unproductive views of proof (Ko, 2010). In response, several university-based teacher preparation programs attempted to support future teachers' competencies to enact reasoning and proving in classrooms (e.g., Buchbinder & McCrone, 2020; Conner et al., 2014). Other researchers suggest that the barriers to the integration of reasoning and proving in classrooms are rooted in sociocultural factors inherent in the institutions of schooling (Chazan et al., 2016). Teachers need to navigate a complex system of often competing expectations by various stakeholders: students, parents, and administration. In such an environment, a teacher's desire to enact reasoning and proving is weighed against a "need to cover" the curriculum (Kotelawala, 2016).

Practical Rationality and Four Professional Obligations

The theory of *practical rationality* (Herbst & Chazan, 2003, 2011) suggests that becoming a mathematics teacher involves adopting a particular framework for making decisions about classroom teaching. This framework involves balancing four types of professional obligations. The obligation to the *discipline* of mathematics involves accurately representing mathematical concepts and engaging students in meaningful mathematical practices, such as reasoning and proving. The obligation to students as *individuals* involves attending to fairness and catering to each student's cognitive and emotional needs. *Interpersonal* obligation considers the class as a whole, requiring the teacher to manage social dynamics and intergroup relations to ensure fair sharing of resources. The *institutional* obligation requires the teacher to follow school, district, and state policies related to curriculum assessment and standards, and adhere to practices and guidelines shared by members of school mathematics departments. While obligations do not prescribe actions, they collectively influence how teachers navigate their professional roles, regardless of their personal beliefs or competencies (Chazan et al., 2016).

We draw on the theory of *practical rationality* and the four professional obligations to explore how Nancy navigated different spaces—her university-based teacher preparation program, supervised internship, and autonomous teaching —and how she balanced various obligations while developing her own teaching practices around reasoning and proving.

Method

Nancy was one of eight beginning secondary mathematics teachers whom we followed for four years as a part of the larger, NSF-funded longitudinal project, and for whom we compiled the most complete data set. Nancy had a strong mathematical background and expressed productive dispositions toward reasoning and proof, making for a rich case study (Yin, 2003).

Data Collection and Analysis

As a preservice secondary mathematics teacher (PST), Nancy took a specially-designed capstone course *Mathematical Reasoning and Proof for Secondary Teachers* (Buchbinder & McCrone, 2020, 2023). The course focused on four proof themes—topics considered in the literature particularly challenging to learn and teach: direct proof and argument evaluation, conditional statements, quantification and the role of examples in proving, and indirect reasoning. This course supported PSTs' developing proof-specific mathematical knowledge for teaching by providing opportunities to crystalize their knowledge of the four proof themes; connect it to students' conceptions of proof and apply this knowledge by designing and teaching in local schools four proof-integrated lessons. Of all the data sources, for this paper, we focused on analyzing Nancy's written reflections about teaching in the course.

An internship is a one-year, post-graduate, supervised teaching experience. Nancy was present full-time in the classroom of her mentor teacher (MT), gradually expanding the scope of her classroom responsibilities. The data for this period included five lesson observations, related instructional materials, and follow-up debrief interviews with a researcher on the project.

As a novice teacher (NT), Nancy moved to another state across the country, so data were collected through online interviews (two in year 1 and three in year 2). Nancy also participated in the Professional Learning Community on Reasoning and Proving (PLC-RP), led by the first author of this paper. The PLC-RP met online, four times a year for 1.5 hours. The video recordings of individual interviews and the PLC-RP meetings comprised the main data sources for the NT period, supplemented by lesson plans and sample student work.

Nancy's transcribed interview recordings from INT and NT years and the written report from her PST period were segmented into thematic units of varied length (from a single utterance to ~10 sentences). Each segment was analyzed to identify Nancy's actions or decisions and her justifications for these decisions. The justifications were examined to discern underlying obligations according to Chazan et al.'s (2016) framework of four professional obligations. Subcategories for each obligation were developed through open coding and the constant comparative method (Corbin & Strauss, 2015) based on Nancy's specific circumstances, e.g. her interactions with her MT during the internship reflect how she upheld the institutional obligation. Some actions were justified not by obligations but by Nancy's personal beliefs and values reflecting her emerging educational approach. We illustrate these in the results section.

Results

The PST Period

As a PST, Nancy did not have to attend to institutional obligations – her sole focus was on developing her mathematical and pedagogical knowledge and experimenting with proof. Nancy shared, "before taking this [capstone course], I only associated proofs with high school geometry and didn't realize all the different proof topics that can be integrated into a [secondary] math classroom." While Nancy did her course practicum in a high-school geometry class, she felt that the course allowed her to develop a more open approach to proof, saying, "I've learned […] a number of ways to integrate these [proof] topics into math classrooms." Illustrating how she integrated the proof themes into her lessons, Nancy wrote:

For example, with conditional statements, I had the students investigate the P & Q, the truth value, and the converse and its' truth value of conditional statements about triangles and segments in triangles since that was what they were learning at the time.

Nancy felt that direct proof and conditional statements proof themes could be "seamlessly" integrated into her lesson plans compared to quantification and role of examples in proving, and indirect reasoning, which she described as "a little forced." Still, Nancy found creative ways to integrate these proof themes into her lessons. For example, for the lesson on indirect reasoning[1], Nancy designed *The Quadrilateral Detective* game, where students used analytic geometry to determine the type of quadrilateral from the given coordinates. Students applied indirect reasoning by writing statements of the form "A quadrilateral cannot be __because__" (e.g., "cannot be a square, because not all sides are congruent") (Liu & Buchbinder, 2022).

Reflecting on the course, Nancy described that it expanded her view of proof and provided her with experiences she wanted to enact in her future classrooms. She wrote:

My teaching philosophy is centered around exploration and reasoning, but before [the course] I wasn't really sure how to enact it in my classroom. […] This course challenged me to think about proving in new ways and helped me to see its benefit in all math classrooms [and] gave me confidence in my abilities.

This quote, while containing no action, shows Nancy's emergent commitment to disciplinary values - "exploration and reasoning." She attributed to the capstone course her increased understanding of what integrating reasoning and proving in mathematics classrooms means, and increased willingness and confidence to do so in her own classrooms.

The Internship (INT) Period

Nancy's internship was with an experienced 8th-grade MT, who shared Nancy's commitment to fostering student mathematical engagement. However, they both found it challenging to enact these values due to the institutional constraints – a rigid curriculum. Nancy appreciated that "the curriculum is good about doing reasoning and proof stuff," but it came in the form of "a booklet that tells us all the lessons that we need to do, and [the] estimated times for everything." To address this, Nancy and her MT redesigned lessons to make them more rich. One example was Nancy's *Halloween Scavenger Hunt*–a Halloween-themed activity that she designed to replace a mundane exam review packet on transformational geometry. Working in groups, students solved problems worded as riddles hidden around the room; each solution led them to a new problem location and a clue to a combination lock box containing a prize. But, except for one problem involving indirect reasoning ("Of the four given sequences of transformations, identify one that does *not* result in transforming a shape FRANK into F'R'A'N'K'"), all other problems focused on calculations since Nancy's goal was for students to "enjoy doing some practice problems."

Thus, Nancy's focus was on the obligation to individual students' engagement, while the disciplinary obligation to reasoning and proving moved to the background. When asked directly about this, Nancy explained that "eventually students are going to prove […], but right now, day to day, we're doing a lot more reasoning than we are doing proving." She also intentionally tried "to ask a lot of *why* questions because it gets them thinking about justifying and explaining."

Compared to Nancy's PST period, this might seem like lowering the bar for integrating reasoning and proving. However, the institutional context is critical to this change. While Nancy enjoyed the support of her MT to follow her aspirations, she was constrained by the institutional

[1] We refer to indirect reasoning as any reasoning of the form "it cannot be so because otherwise", whereas indirect proof is a proof by contradiction or contrapositive (Yopp, 2020).

obligation to adhere to her school's rigid curriculum and procedures. Thus, sustaining a classroom culture where students are actively engaged in "fun" activities became Nancy's primary goal, a pre-condition for enacting her "teaching philosophy centered around exploration and reasoning." This is evidenced in her repeated use of "why does it work?" questions, supporting students in "justifying and explaining" and "understanding mathematics as a whole." She also used self-designed problems to substitute or supplement the curriculum. Nancy's approach reflects the tensions between her own values, aligned with individual and disciplinary obligations, and the obligation to meet the institutional standards of her internship school.

The Novice Teaching (NT) Period

Nancy started her first year as a high school teacher in a different US state. In the spring of her first year of autonomous teaching, Nancy described tensions between her desire to support students' mathematical learning and the need to adhere to school standards and curriculum, saying, "the standards and the things that [the school] wants [teachers] to teach are taking away from some of the fun stuff." Nancy also shared her frustration with some of the pedagogical approaches of her colleagues, especially the teacher with whom she was expected to coordinate. Nancy said: "Basically what she [the teacher] does is […] walks through guided notes for 30 minutes and then gives them [students] practice problems for 30 minutes. And that kills me!"

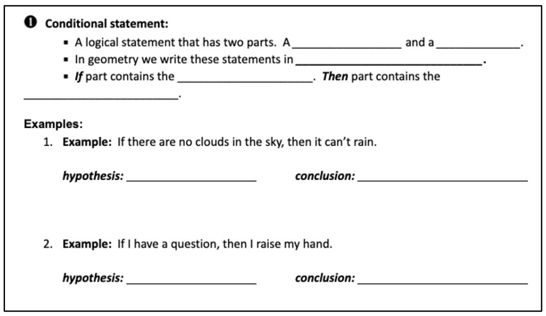
Several teachers in Nancy's school shared her desire to "move away from the guided notes" toward a "more student-focused curriculum" but met with opposition from other teachers whom Nancy described as "old school," clinging to "old traditional ways." Despite the institutional challenges: the divisive school culture, traditional teaching practices of her direct colleague, and teacher-centered instructional resources "handed to her," Nancy strived to teach her classes in student-centered ways, aligned with her teaching philosophy, even at the cost of extra work hours. Her persistence paid off in her second year when she developed her *Proof Unit*.

1. Introduction to Proof Concepts ❑ PowerPoint defining conjectures, indirect and inductive reasoning and counterexamples. ❑ Interactive Activities: SET Rules, Snowflake Activity ❑ Practice with patterns and counterexamples	2. Conditional Statements and Logical Puzzles ❑ PowerPoint presentation on vocabulary: conditional statements, converses, contrapositives and inverses. ❑ Practice creating conditional statements ❑ Interactive Activity: Sudoku	3. Proving the Area of a Circle ❑ Proof 1: Rearranging sectors of a circle to form a parallelogram and a rectangle (hands-on task) ❑ Proof 2: Unfolding concentric rings to form a triangle
4. Combinatorial Problems and Game Strategies ❑ Video introducing the "Game of 100" ❑ Movement activity on game strategies ❑ Introduction to the "Handshake Activity" for combinatorial problems	5. Combining Mathematical Concepts and Problem-Solving ❑ Continuation of the "Handshake Activity" ❑ Triangle Inequality Theorem Exploration ❑ Fibonacci Rectangles Exploration ❑ Sudoku puzzle for Problem-Solving	6 & 7. Summative Assessment ❑ Poster Project: students work individually to create a poster summarizing and expanding on one of the proof activities from the unit. ❑ Followed by a gallery walk.

Figure 2. Nancy's Proof Unit outline.

Nancy's Proof-Unit. The 11th-grade geometry new state-wide curriculum standards called for a unit on "reasoning and proof." However, there were no pre-existing curriculum materials. As a result, the school allowed the teachers to use curriculum materials of their choice and relaxed the usual requirement for teachers to coordinate assessments across classes. In Nancy's words, this was a "rare situation." With this relaxed institutional pressure, Nancy jumped on the opportunity to integrate her desired teaching practices, saying: "I got this! I have all these really great ideas, I have the PLC where we do proving and reasoning, like, I can come up with this really cool thing!" Undeterred by the lack of her colleagues' collaboration, Nancy decided to "pull from the proofs […] that we've done [in the course and the PLC] to do some of the cooler parts of math." The outcome was a three-week unit (Figure 2) where each lesson involved "different activities about reasoning and proving." For the summative assessment, Nancy had each student create a poster that summarized and extended one of the unit's proof activities.

Nancy utilized many activities from the PLC-RP (e.g., proving the area of a circle) and the capstone course, like introducing proof-related (e.g., conjectures, patterns) and logic-related concepts (e.g., conditional statements, contrapositive) using both mathematical and real-world examples (Figure 3a). She started by having students explore visual and numeric patterns, introducing them to conjecture, inductive reasoning, and counterexamples (Figure 4).



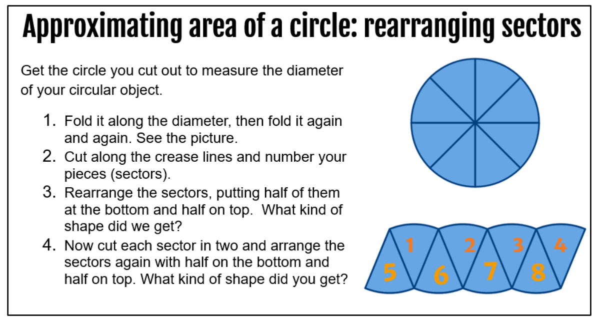


Figure 3a (on the left). A slide introducing conditional statements using non-mathematical examples. Figure 3b (on the right). A slide from the "proving the area of a circle" activity (lesson 3).

Nancy explained that the first two lessons in the unit "weren't necessarily proof, but they were just like, things that have to do with proving." The rest of the lessons involved what Nancy called "a bunch of different mini-proofs" – pattern explorations and informal arguments. For example, describing lesson 3 on proving the circle area formula (Figure 3b), Nancy said:

> [We] did the proof of the area of a circle, where the students have a circle and then they change it into a parallelogram […] That was kind of a fun day because […] I had them go through the exercise of folding [the circle] and cutting it and laying it out on their desk […] So, they actually had hands-on proof that they engaged in the entire time.

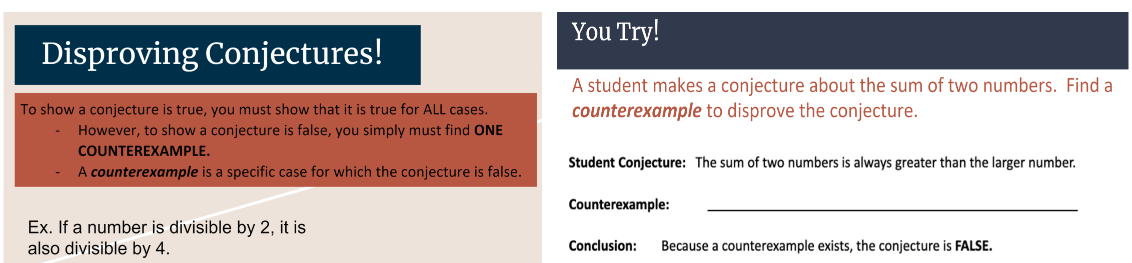


Figure 4. Explanation of the meaning of a counterexample and a task for students to do on their own.

These examples show that Nancy's focus was not on writing formal proofs, but on having them visualize, generalize patterns, formulate conjectures, and justify their thinking. This aligned with her disciplinary obligation to introduce mathematical reasoning and proving while considering students' minimal prior knowledge (individual obligation). The individual obligation is also evidenced in the focus on students' active engagement through games, hands-on tasks, and explorations. Nancy worked to make reasoning and proving enjoyable and appealing.

Takeaways and community reactions. Nancy enacted her Proof Unit with two lower-level geometry sophomore classes. She was pleased with the outcomes, especially when some students took the initiative to explore concepts beyond class notes. Nancy's feelings were apparent in her voice and discourse: in a 49-minute interview, she used the words "cool" 30 times and "excited" 15 times. Encouraged by the positive experience, Nancy planned to revise the unit next year, to further emphasize proof, which reflects her increased attention to the disciplinary obligation.

Returning to institutional obligation, Nancy shared her unit and student outcomes with the administration and other teachers. She felt that many of her colleagues "were inspired," but some had reservations, motivating them by the need for consistency in teaching. Nancy, apparently, refused to be constrained by this. She said: "The big takeaway for me is - try new things even if other people aren't on board. Still do it, and you might get some really good outcomes from it."

Summary and Discussion

Putting Nancy's experiences in the broader context of preparing mathematics teachers to enact reasoning and proof in schools, we ask: What is Nancy a case of? Nancy is a well-prepared teacher candidate (AMTE, 2017) with robust mathematical knowledge, a strong pedagogical orientation toward active learning, and a commitment to engaging students with reasoning and proof—ideas she cultivated in the capstone course (Buchbinder & McCrone, 2023). As a PST, Nancy was shielded from the institutional obligation, enabling her to focus on mathematics, students' interactions with it (i.e. disciplinary obligation), and her own professional growth. As an INT, Nancy faced institutional constraints, including a rigid curriculum and a quick teaching pace. However, with her MT's support, Nancy was able to design and enact some reform-oriented activities to uphold her obligation to student engagement and to the discipline, which Nancy described as "interactive things and exploratory things and proof-based things," like her *Halloween Scavenger Hunt.* This illuminates the importance of placing INTs with mentors whose teaching philosophy aligns with that of university teacher preparation programs, and who can support rather than constrain INTs' professional aspirations (Gainsburg, 2012; Thompson et al., 2013), in this case – Nancy's "teaching philosophy centered on exploration and reasoning".

As an NT, Nancy faced increased institutional constraints and found it even more difficult to balance them with disciplinary and individual obligations. The need to coordinate assessments with other teachers, some of whom had vastly different pedagogical approaches from Nancy's, created tension for her. This is consistent with Wang et al., (2008) and Feiman-Nemser's (2003) findings on the influence of school culture on the nature and quality of the early years of teaching practice. Despite this, Nancy remained determined to teach in the way she saw as most advantageous to student learning. She utilized activities from the capstone course ("I did pull a couple of things from my time [in the capstone course]") to integrate "exploratory" and "proof-based activities." Nancy's nonconforming attitude and sense of agency are possibly unique to her personality. However, what is not unique is the fact that she had strong mathematical and pedagogical foundations, as well as access to instructional resources and continued professional support around reasoning and proof, aspects that have been identified in the literature as positively affecting novice teachers' practices (Wang et al., 2008; Vescio et al., 2008). Nancy's case illustrates that despite the challenges, novice teachers can draw on their professional preparation to develop rich opportunities for student learning of reasoning and proving.

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References

Association of Mathematics Teacher Educators (AMTE). (2017). *Standards for Preparing Teachers of Mathematics*. Available online at: amte.net/standards.

Bieda, K. N., Sela, H., & Chazan, D. (2015). "You are learning well my dear": Shifts in novice teachers' talk about teaching during their internship. *Journal of Teacher Education, 66*(2), 150-169.

Buchbinder, O., & McCrone, S. (2020). Preservice teachers learning to teach proof through classroom implementation: Successes and challenges. *Journal of Mathematical Behavior, 58*, 100779.

Buchbinder, O., & McCrone, S. (2022). Guiding principles for teaching mathematics via reasoning and proving. *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education.* Bozen-Bolzano, Italy.

Buchbinder O., & McCrone, S. (2023). Prospective secondary teachers' learning to teach mathematical reasoning and proof: The case of the role of examples in proving. *ZDM – Mathematics Education, 55*(5), 779–792. https://doi.org/10.1007/s11858-023-01493-4

Chazan, D., Herbst, P., & Clark, L. (2016). Research on the teaching of mathematics: A call to theorize the role of society and schooling in mathematics instruction. In D. Gitomer, & C. Bell (Eds.), *Handbook of research on teaching* (5th ed., pp. 1039–1097). American Educational Research Association.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401-429.

Corbin, J., & Strauss, A. (2015). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (4th ed.). Sage.

Ellis, A. B., Bieda, K., & Knuth, E. (2012). *Developing Essential Understanding of Proof and Proving for Teaching Mathematics in Grades 9-12*. NCTM

Feiman-Nemser, S. (2003). What new teachers need to learn. *Educational leadership*, *60*(8), 25-29.

Gainsburg, J. (2012). Why new mathematics teachers do or don't use practices emphasized in their credential program. *Journal of Mathematics Teacher Education*, *15*(5), 359–379.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the learning of Mathematics*, *23*(1), 2-14.

Herbst, P., & Chazan, D. (2011). Research on practical rationality: Studying the justification of actions in mathematics teaching. *The Mathematics Enthusiast, 8*(3), 405–462.

Ko, Y. Y. (2010). Mathematics teachers' conceptions of proof: Implications for educational research. *International Journal of Science and Mathematics Education, 8*(6)*,* 1109-1129.

Kotelawala, U. (2016). The status of proving among US secondary mathematics teachers. *International Journal of Science and Mathematics Education, 14*(6), 1113-1131.

Liu, J ., & Buchbinder, O., (2022). Learning to Teach Reasoning and Proof in an Online Setting: The Case of Nancy. In Karunakaran, S. S., & Higgins, A. (Eds.) *Proceedings of the 24th annual Conference on Research in Undergraduate Mathematics Education*, (pp. 357–365). Boston, MA.

Reid, D., & Knipping, C. (2010). *Proof in Mathematics Education: Research, Learning and Teaching.* Sense Publishers.

Stylianides, G. J., Stylianides, A. J., & Shilling-Traina, L. N. (2013). Prospective teachers' challenges in teaching reasoning-and-proving. *International Journal of Science and Mathematics Education, 11*(6), 1463–1490.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 237-266). NCTM.

Thompson, J., Windschitl, M., & Braaten, M. (2013). Developing a theory of ambitious early-career teacher practice. *American Educational Research Journal*, *50*(3), 574-615.

Wang, J., Odell, S. J., & Schwille, S. A. (2008). Effects of teacher induction on beginning teachers' teaching: A critical review of the literature. *Journal of Teacher Education*, *59*(2), 132-152.

Wasserman, N., Buchbinder, O., & Buchholtz, N. (2023). Making university mathematics matter for secondary teacher preparation. *ZDM – Mathematics Education,55*(5), 719–736. https://doi.org/10.1007/s11858-023-01484-5

Vescio, V., Ross, D., & Adams, A. (2008). A review of research on the impact of professional learning communities on teaching practice and student learning. *Teaching and Teacher Education*, *24*(1), 80-91.

Yin, R. K. (2017). *Case Study Research and Applications: Design and Methods* 6th Edition. Sage publications.

Yopp, D. A. (2020). Eliminating counterexamples: An intervention for improving adolescents' contrapositive reasoning. The Journal of Mathematical Behavior, 59, 100794.