Transformational Reasoning in an Introduction to Proofs Course

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The goal of this preliminary report is to demonstrate students engaging in transformational reasoning in an introduction to proof class. Utilizing transcript data and operationalizing Simon's (1996) definition of transformational reasoning, these preliminary results include descriptions of transformational reasoning in an advanced math context, as well as how students used transformational reasoning in order to accomplish their mathematical goals.

Keywords: transformational reasoning, introduction to proof

Inquiry-oriented instruction (IOI) provides a classroom environment in which students can share and continually refine their thinking. In IOI the classroom mathematical agenda is advanced by building on students' informal ways of reasoning to develop more formal mathematical concepts. One such type of informal way of reasoning that provides a powerful foundation for more advanced reasoning is transformational reasoning (Simon, 1996). Transformational reasoning involves (1) mentally (or physically) enacting operations on objects which result in the objects being transformed and (2) observing the result of the operation, which results in one coming away with a sense of how something works (Simon, 1996).

Scholars have studied transformational reasoning in a variety of math contexts. These include pre-calculus and calculus (Carlson et al., 2002; Johnson, 2012), algebra and geometry (Ellis, 2007; Simon, 1996; Zandieh et al., 2008), and proof and justification (Harel & Sowder, 1998).

Theoretical Perspective

Simon (1996) coined the term transformational reasoning to describe a type of mathematical reasoning that he noticed students were spontaneously engaging in, that resulted in the students coming away with a sense of how something worked. Simon defines transformational reasoning in the following way:

> Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider not a static state, but a dynamic process by which a new state or a continuum of states are generated. (1996, p. 201)

This definition includes (a) an object (or objects) that is (are) being transformed, (b) the operation(s) that affect the object (or objects), and (c) results of the operation(s) on the object(s). The process of transformational reasoning involves identifying (a) and (b), which allows one to envision the transformation taking place, and once (c) is attained one can examine them together to draw conclusions about the entire process to understand why it works.

To get a better feel for transformational reasoning, consider the *bisection method*. The idea of the bisection method is to generate (via bisection) a sequence of intervals of decreasing length. One application of the bisection method is to approximate roots of continuous functions with a sign change. In this application, the intervals all share the property that the function outputs have opposite signs at the endpoints, i.e., positive output on the left and negative output on the right or vice versa. The sequences of the right and left endpoints converge to a root of the function.

One might engage in transformational reasoning in this context by focusing one's attention on the intervals as defined above, i.e., the object. The operation is that the intervals decrease in length at each step. The result is increasingly small intervals, the lengths of which converge to zero. The resulting sense for how the bisection method works comes from realizing that the interval length shrinking to zero means that the endpoints will match, and since the function is continuous these endpoints must be a root.

As noted above, it is critical that students consider a dynamic process in order to engage in transformational reasoning. Simon goes on to state that this dynamic process could be imagined or concrete (Simon, 1996, p. 202). Since transformational reasoning results in figuring out how something works, Simon argued that this was different from following a chain of reasoning (i.e., deductive reasoning) or noticing and describing a pattern (i.e., inductive reasoning) (p. 197) but that it may overlap with either deductive or inductive reasoning depending on the circumstances (p. 204).

Transformational reasoning supports the ascertainment of evidence when one is proving. Harel and Sowder (1998) report on a series of proof schemes that explain the ways in which students might engage in proving, one of which is the transformational proof scheme. For the authors a proof scheme "consists of what constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998, p. 244). Thus, a proof scheme contains two parts: (1) how an individual convinces themselves (ascertaining), and (2) how an individual convinces someone else (persuading). Convincing oneself necessitates understanding how something works, and transformational reasoning can support this. Therefore, transformational reasoning supports ascertainment of evidence regardless of the math activity in which one is engaging.

In the present study, students in an introduction to proof class were engaged in the guided reinvention of fundamental real analysis concepts and I explored the students' activity by investigating two research questions: (1) what does transformational reasoning look like in an inquiry-oriented introduction to proof class? and (2) how do the students use transformational reasoning to accomplish their math goals?

Methods

Data for this study come from a large NSF-funded design project ASPIRE in Math (DUE# 1916490). Briefly, the ASPIRE in Math project sought to support students in transitioning to advanced math while also supporting instructors to shift to inquiry-based mathematics teaching. A whole class teaching experiment, in the form of an introduction to proof class, was conducted in Spring 2020 at a public university in the Pacific Northwest of the United States during the beginning of the COVID-19 pandemic. The goal of the teaching experiment was to finalize a curriculum and to begin articulating instructor support materials. The goals of the curriculum included engaging students in the guided reinvention of fundamental concepts in real analysis and developing advanced math practices including defining, conjecturing, and proving. The task sequence for the curriculum involved students proving the Intermediate Value Theorem by repurposing the bisection method for approximating a root of a continuous function (see Larsen et al., 2022).

Each of the 20 class sessions lasted 110 minutes. The research team included a professor of mathematics education, and two doctoral students (one of which is the author of this report). Each class was captured using Zoom recording and screen recording technology. Other data collected included student generated artifacts from various tasks that were recorded on Google Docs, slideshows, and Jamboards. The students (all names are pseudonyms) would record conjectures, informal definitions, informal proofs, and images created or screenshots in these documents.

Data analyzed for this preliminary report came from transcripts of two different class days, and associated Google Docs. These two days were selected because of the content of the mathematical discussions. The episodes analyzed include students identifying sequences generated by the bisection method, students conjecturing about properties of the sequences, and students justifying that the left endpoint sequence converges.

I conducted a two-phase analysis. In the first phase I sought to ascertain and describe instances of transformational reasoning in terms of the components (i.e., objects, operations, results) and the resulting “sense of knowing how things work”. Within student utterances I coded for (1) objects that make up a model (Lesh et al., 2000), (2) operations that transform the objects in the model, (3) observations about the outcomes of the operations, and (4) mentions of structural features (e.g., referring to a sign change, knowing that the midpoint becomes an endpoint of a subsequent interval, etc.) of the model. Structural features were used to identify when students were primarily explaining the process for implementing the bisection method. The second phase involved analyzing how students used this reasoning to address their mathematical goals.

Initial Results

Students used transformational reasoning to identify sequences, conjecture about sequences, and justify that the left endpoint sequence converges.

Identifying Sequences Generated by the Bisection Method

The class session began with a demonstration of how the bisection method can be seen to generate sequences, by showing how the left endpoint sequence is generated, followed by formally defining *sequence*. Then the students were invited to come up with as many sequences as they could (see figure 1).

1. Sequence of left-endpoints, a_n.
2. Sequence of right-endpoints, b_n.
3. Sequence of function values at c_n.
4. Sequence of midpoints, c_n.
5. Sequence of $f(a_n)$ values.
6. Sequence of $f(b_n)$ values.
7. Sequence of step numbers.
8. Sequence of the number of digits at each step in the midpoint.
9. Sequence of $b_n - a_n$
10. Sequence of the distance c_n is from the root (sequence of errors).
11. Sequence of number of matching digits after each iteration.

Figure 1. List of sequences identified by the students in class

Leo offered sequence 8 (see figure 1). Leo’s explanation showed signs of transformational reasoning. He stated that “when you do the bisection method, you add, decimals are added at each step, and if you have a calculator that could keep up, then you could count those.” Here Leo’s object of focus was the midpoint sequence (i.e., the root approximation at each step in the method). I conjecture that the transformation he was envisioning is the computation of the midpoint at each step, and the outcome of this transformation is the addition of new digits in the midpoint at each step. Leo’s description implies that he had a feel for how the midpoint sequence works and that allowed him to identify a sequence that counts the number of digits in the midpoint at each step.

Conjecturing About Properties of the Sequences

Later, students were asked to conjecture any properties about the sequences that they noticed or limits the sequences might have. Maya and her groupmate Kevin worked together to produce a lengthy document with several observations about each sequence. A sample is provided in figure 2.

1. Sequence of left-endpoints, a_n.

Can repeat in sequence but cannot go back after you leave the number

Increasing or staying the same

Figure 2. Sample conjectures from Maya and Kevin

Maya and Kevin's description of the sequence behavior indicated that they were engaging in transformational reasoning. Maya and Kevin identified the object they were focusing on in the bisection method as the left endpoint sequence. In the remark "can repeat in sequence but cannot go back after you leave the number", the pair identified an operation that takes place as the left endpoint sequence is generated in one step. I conjecture that the transformation they envisioned is one step in the process, and they observed that the left endpoint sequence either stands still ("repeat") or moves forward, but "cannot go back". Then, as a consequence of this one step, I further conjecture they envisioned generating the entire left endpoint sequence which resulted in an understanding of how the left endpoint sequence works which they encode as either "increasing or staying the same".

Justifying that the left endpoint sequence converges

A few class-days later, the students come to the conclusion that the left endpoint sequence converges, and then they are asked to articulate why this is the case. Maya's explanation builds on her previous transformational reasoning discussed above.

> *Maya*: So, when you're doing this method, each next term of the left endpoint sequence that you're making is either going to be the same as the term you were on because the other one changed, or it's going to go halfway to the other term, right? So um, you're either adding zero to it if it stayed the same, or you're adding the difference between it and b sub the same n.
>
> *Teacher*: And what is the remaining difference?
>
> *Maya*: So, it's $\frac{a_n + b_n}{2}$, or if you're thinking about it in terms of what you're adding to a_n, it's $\frac{b_n - a_n}{2}$, and the limit of that as n goes to infinity becomes zero, so you end up adding smaller and smaller things. So, you know that long run a_n has to stop moving or have a limit.

Maya's use of transformational reasoning results in a justification for why the left endpoint sequence converges. She begins by utilizing the transformational reasoning she arrived at in the previous section. However, Maya was not imagining the formation of the left endpoint sequence in the same way. Her description was more specific. It went from “can repeat in sequence but cannot go back” (see figure 2) to “it’s either going to be the same as the term you were on… or it’s going to go halfway to the other term”. Then, since she had a feel for how the left endpoint works, she transitioned to focusing on a new object that is within the left endpoint sequence, namely what gets added onto the previous term: 0 or $\frac{b_n - a_n}{2}$. I conjecture that the transformation she envisions is that the terms in the left endpoint sequence are growing because they are accumulating small amounts. The result of which is that the sequence converges because the amounts that are accumulated become smaller and smaller. By understanding how the left endpoint sequence works, she was able to home in on the feature of the sequence that explained her intuition that the sequence converges.

After this activity, students were asked to generalize their justifications into conjectures about general sequences. This was done by identifying the properties that were used to justify their thinking and written in the form “If [property or properties] then x_n converges.” Then these conjectures were evaluated for plausibility with examples and counterexamples after which they would work to prove some of the conjectures. For example, the class established that Maya's argument could work but only if it is made more specific by comparing the amount "added on" to the terms of a geometric series.

Concluding Remarks

My results demonstrate two main things: (1) what transformational reasoning looks like in an intro-to-proof context, and (2) how students use transformational reasoning to engage in advanced math activity. In terms of the first research question by identifying objects, operations, and results in student utterances, I was able to describe how students came to know how things worked. This method provides a means for other researchers to unpack transformational reasoning in other math contexts.

Engaging in transformational reasoning supported students to accomplish their math goals. Students identified sequences that are generated by the bisection method and conjectured and justified properties that the sequences have. My results indicate that students might use a two-step approach to engaging in transformational reasoning that allows them to advance their ideas further, as in the case of Maya. Maya and Kevin’s two-step process involved first imagining how to go from one term to the next, and then using the results of that reasoning to further reason about the generation of the entire left endpoint sequence. Maya’s justification that the left endpoint sequence converged has a similar structure.

Maya’s activity demonstrated that transformational reasoning could support the identification of key ideas (Raman, 2003, 2004). Maya began by articulating how the left endpoint sequence worked. Once she knew it had to converge, she used her sense of understanding to ascertain *why*. The key idea for Maya what gets added on to the previous term converges to 0.

The instructional materials used in the teaching experiment here also engage students in algorithmatizing, defining, and proving general conjectures. The next phase of my analysis will be examining these other mathematical activities for instances of students engaging in transformational reasoning. The presentation will include an expanded data set.

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