An Analysis of the Conceptual and Authentic Opportunities in Calculus I Textbook Tasks

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Gatekeeper mathematics courses, including calculus, historically prevent marginalized students from pursing STEM majors by often focusing on procedural fluency over conceptual depth. Efforts to better prepare students for future career pathways include using tasks that emphasize authentic application contexts. In this study, we analyzed application tasks within the first half of the derivatives unit for two commonly used undergraduate Calculus I textbooks. Our analysis identified both the authenticity of the application context, and the conceptual understanding required to solve each task. By juxtaposing these task features, we found notable discrepancies between the degree to which tasks presented conceptual opportunities within authentic, real-world applications. These preliminary findings inform the need for further research investigating the relationship between conceptual understanding and authenticity in application tasks. This work can inform the design of novel tasks which appeal to both features, thus better preparing students to succeed in both calculus and future degree pathways.

Keywords: calculus, textbooks, authentic applications, conceptual understanding

Introduction

Entry-level collegiate mathematics courses, such as Calculus I, often function as gatekeeper courses because they are historically documented to prevent students marginalized by their racial and/or gender identities from pursuing STEM majors (Battey et al., 2022; McNeill et al., 2022; 2023; Leyva et al., 2021). Considering the importance of such courses for students' success in a variety of career pathways, there is a need to investigate approaches to make such courses mathematically rich, contextually relevant, and accessible. One approach to address this problem is to understand if tasks within these courses present opportunities for conceptual mathematical engagement within authentic applications (i.e., tasks with a real-life context where a concept from a mathematical course is being used in a way that is relevant to that context). While there have been studies about the opportunities for conceptual learning available to students in gatekeeper courses such as Calculus I via textbooks (Palm et al., 2006; Tallman et al., 2021), there has not been an analysis that identifies those opportunities as presented within authentic applications. Identifying if those opportunities exist, even just within textbooks, can impact whether undergraduate students engage with mathematics concepts as relevant within contexts outside the mathematics classroom. Furthermore, such an analysis can shape curriculum writing and assessment development to address the problem of gatekeeper courses. Therefore, this paper attends to this problem by asking the following research question: *In problems available within the derivative lessons of Calculus I textbooks, how do the tasks in textbooks compare in terms of conceptual understanding within tasks with real-life applications?*

Literature Review

Much of the existing research on gatekeeper courses focuses specifically on calculus, which is often considered to represent the thinking necessary to thrive in ambitious, STEM-forward

careers. The body of literature suggests different interpretations of what is considered a relevant and meaningful task for students within the calculus context. First, calculus courses vary in terms of which standards are covered and how those standards are sequenced, which has led to high attrition rates (Barr et al., 2022; Schmidt et al., 2022). Second, among those standards covered, calculus courses often focus on learning outcomes that exceedingly prioritize procedural accuracy (NCES, 1995). Both structural elements of calculus courses contribute to *weeding out* students who are assumed unable to thrive, demonstrating the current role of calculus as a gatekeeper course (Ellis et al., 2015; Larsen et al., 2016; Leyva, 2021). Finally, although calculus courses are deemed critical steppingstones for students to access STEM careers, there is evidence that minoritized groups are underrepresented in the enrollment and completion of such courses (Adelman, 1999), and that the coursework does not include clear connections to students' interests or lived experiences (Adiredja, 2021; Alzaga Elizondo & Larsen, 2024; McNeill et al., 2022). Together, the combination of inconsistent content standards and mathematical expectations leads to course-wide structural decisions that, intentionally or unintentionally, push students out. This, alongside evidence of the under-enrollment of minoritized students, suggests a need to intervene in how calculus concepts are shared.

One place to intervene is to investigate whether the mathematical tasks available in gatekeeper courses can challenge students' prior experiences with conceptual and application-based tasks. In elementary and secondary mathematics classes, students develop habits to find and complete computational steps on tasks that include applications without considering or acknowledging the context provided (Schoenfeld, 1989). After secondary mathematics courses, students further develop habits to ignore how the numbers, units, and operations within real-world problems conceptually relate to the challenge presented within context (Reinke et al., 2024; Schoenfeld, 1989; Verschaffel et al., 2000). Even the intention to interweave conceptual ideas to real life applications can be irrelevant if the tasks can be solved by only attending to the associated procedural computations. Although not a complete solution, understanding what tasks are commonly available for professors to assign students provides an initial step toward mitigating courses like Calculus I from gatekeeping students.

Methods

To identify opportunities for students to engage in conceptual understanding within authentic applications, we focused on the sections introducing one core calculus concept, derivatives, across two textbooks. We selected the textbooks by surveying publicly available syllabi to create a sample which reflects different approaches used across calculus courses, being a traditional textbook (Rogawski, 2011) and a problem-based textbook (Boelkins, 2013). For the analysis, we examined problems specifically within sections that (1) introduced to the derivative and (2) introduced to the concept of the derivative as a function. To select problems to answer our research question, we examined whether problems included text outside of the expression, equation, or directions that was situated in some real-life context. We adapted two frameworks (Greene & Shorter, 2017; Palm, 2008) into deductive coding schemes (Fereday & Muir-Cochrane, 2006) to measure conceptual richness and authenticity, which were used to assign each for conceptual rigor, based Greene & Shorter's (2017) conceptual understanding weighting system (CUWS), and another for the authenticity of the application, based on Palm's (2008) research on authentic applications (AA).

To rate a task's conceptual understanding (i.e., CUWS score), a task could earn a point for the *skills* required (i.e., if the task did not involve only computational or algebraic skills or memorized facts, and required understanding of the concept), the *method* required (i.e., if the

task could not be answered completely using a method that the student might have developed in prior coursework), and the *conceptual rigor* required (i.e., if the task involved interpreting the meaning of mathematical characteristics, making connections between different representations of mathematical characteristics, and/or proving or justifying mathematical statements), with a maximum of 3 points available. To rate the authentic application (i.e., AA score), a task could earn a point for each of the following: the *event* (i.e., if the real-life scenario described in the task was an actual event, and not a statement about a relationship between two variables in a context), the *question* (i.e., if the question inherent to the task was one that might be reasonably asked outside of the classroom), the *purpose of the task content* (i.e., if answering the question about the event would provide clarity regarding the event), the *language use* (i.e., if the language specific to the event did not hinder answering the question), and *the existence, realism, and specificity of data* (i.e., if the information provided contained realistic values that could be considered close to corresponding values in real life), with a maximum of 5 points available.

Each task was coded by the first and second author in three stages: first, using the original adaptations of the two frameworks for each task; second, adapting the framework based on any discrepancies identified by the coders (Campbell et al., 2013); and third, to complete consensus coding (Watts & Finkenstaedt-Quinn, 2021).

Preliminary Findings

Between the two analyzed textbooks, there was a difference in the prevalence of tasks with some application context. For the traditional textbook, 15 out of 198 tasks examined (7.5%) included an application context; for the problem-based textbook, 29 out of 66 tasks examined (44%) included an application context. After analyzing the conceptual richness and the authentic application of each task including some application context, we additionally found differences in the average CUWS and AA scores between both textbooks (Table 1). First, the central tendency descriptions of the CUWS scores for tasks within each textbook varied, with the traditional textbook having a mean and median CUWS score in the lower third range and the problem-based textbook in the upper third range. However, there was not a noticeable difference in the mean and median AA scores. This suggests that both textbooks presented similar opportunities for authentic applications, but with varying conceptual richness between textbooks.

Table 1. Analysis of Derivative Tasks in the Traditional and Problem-Based Calculus I Textbooks

		CUWS Score		AA Score		Absolute Difference	
Textbook	Number of Tasks	Mean	Median	Mean	Median	Mean	Median
Traditional	15	1.27	0	3.33	3	2.27	3
Problem-Based	29	2.52	3	3.58	4	1.31	1
Difference		1.25	3	0.25	1	0.96	2

Contrasting Characteristics: Absolute Difference between CUWS and AA Score

The differences most pronounced in the average absolute difference between a task's CUWS and AA score (Table 1). This value emerged as we collected the combinations of CUWS and AA scores, noticing that the range between one score and another indicated an important variation between problems that happened to have the same CUWS score (for instance) and a different AA score (and vice versa). In other words, the absolute difference denotes if there exists a difference in the intensity of a task's CUWS and AA score, where larger values indicate that a task is lacking in either its conceptual complexity or its authentic application, despite having a stronger score in the other category. In what follows, we illustrate the differences in the contextual richness of two tasks that cover similar Calculus I content with a similar degree of authenticity.

The two tasks described represent the variation possible in CUWS score for tasks with the same AA score. One of the tasks is drawn from each textbook; the tasks were in similar sections in which the derivative had been defined and introduced but not yet calculated. The task in the traditional textbook, Task A, had a 0-3 score (CUWS-AA) and asked students to use the symmetric difference quotient (SDQ) to approximate the derivative at a certain point within the context of vapor pressure and water temperature. The task in the problem-based textbook, Task B, had a 3-3 score (CUWS-AA) and focused on the US census in Grand Rapids, MI, asking students to compute the average rate of change over an interval that gets smaller. Task A was given a CUWS score of 0 because the task only involved computational and algebraic skills; the task provided the formula to compute the SDQ as a more accurate way to estimate the derivative than the difference quotient, and the context of vapor pressure was irrelevant as long as the student used the given temperature values. Task B, on the other hand, was given a CUWS score of 3 because the task could not be completed using only computational skills or a method from a prior class; instead, the task required students to interpret the meaning of the derivative within the context of population growth. Like Task A, Task B provided formulas to compute the average rate of change and model the population as a function over time. However, the task required students to not only identify the average rate of change but to situate that change across multiple intervals and describe how their calculations contextualize what might have been happening in Grand Rapids within the different time periods. Despite the differences in CUWS scores, the tasks received similar AA scores because they were situated in authentic contexts in which estimating the derivative would provide useful information about the given application.

Contrasting Characteristics: Likelihood of Particular Score Juxtapositions

Across all tasks identified in both textbooks, we also identified the juxtaposed pairs of CUWS and AA scores (Table 2). In the traditional textbook, the most common juxtaposition found was 0-3 (CUWS-AA). Note that this score was given to 8 (53%) of the tasks analyzed in the traditional textbook; in contrast, for the problem-based textbook, this score was given to only one task analyzed (3%). For the problem-based textbook, there were nearly an equal percentage of tasks that were scored 3-5 and 3-4, for a total of 17 tasks (59%). These scores were less common in the traditional textbook, given to three problems analyzed (20%).

Table 2. Analysis of the Most Common Score Juxtapositions of Tasks in the Traditional and Problem-Based Calculus I Textbooks

CUWS Score	AA Score	Tasks in Traditional (N = 15)	Tasks in Problem-Based (N = 29)	Total
0	3	8	1	9
2	3	2	2	4
3	3	2	4	6
3	4	1	9	10
3	5	2	8	10

Discussion

The preliminary findings from our analysis indicate potentially impactful differences between traditional and problem-based calculus textbooks. In the textbook sections introducing the derivative concept, we observed differences in the overall prevalence of tasks including an application context and the degree to which such tasks (1) required conceptual understanding and (2) lent authenticity to the application context. Our findings additionally suggest that discrepancies often exist, for individual tasks, between the degree of authenticity and the degree of conceptual understanding required. While providing an authentic context can enhance students' perceived relevance of calculus, such tasks must *also* engage students' conceptual understanding to improve student learning. Together, these findings suggest the potential impact of the textbooks that calculus instructors select for teaching their courses. Importantly, including more authentic, conceptually rich tasks can create learning environments where students can both recognize the relevance of calculus for their future careers and engage with tasks that support their conceptual understanding.

In the future, we intend to expand our preliminary analysis to include more textbooks and to encompass additional calculus concepts for a more thorough exploration of calculus tasks. This will include investigating the relationship across tasks between the CUWS score and different components of the AA score (i.e., event, purpose, etc.). We will use the combined framework focusing on both conceptual understanding and authenticity of application to provide a structure for developing new tasks across mathematical domains that are both conceptually rich and authentic. Careful attention to the nature of assessment tasks along theses dimensions will provide one pathway to help transform gatekeeper courses, such as calculus, into *gateway* courses that can support students with achieving success in their chosen career pathways.

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