A Methodology for Mapping Connections Between Persistent Learning Challenges

Joseph Antonides
Virginia Tech

Anderson Norton
Virginia Tech

Rachel Arnold
Virginia Tech

Emma Feinstone
Virginia Tech

Michael Holman
Virginia Tech

Much research within mathematics education is dedicated to understanding the challenges that students often face when encountering new mathematical ideas. Some challenges, called epistemological obstacles (EOs), persist despite research-based instruction designed to target those challenges. The purpose of this report is to describe a methodology for mapping connections between EOs as they emerge during mathematical discussion and problem-solving. We illustrate this methodology through a preliminary analysis of whole-class data collected in an introductory proofs course.

Keywords: Methodology; Epistemological Obstacles; Transition-to-Proof

Understanding the obstacles that students face in mathematics teaching and learning is an essential part of mathematics education research. Much research has focused on identifying and understanding students' cognitive obstacles in mathematics learning (e.g., Smith et al., 1994). Such obstacles are often identified and studied independently, but we have found that many obstacles do not occur independently. Existing research methods, however, do not allow for identifying ways that cognitive obstacles may co-occur. This report provides such a method, allowing researchers to gain greater insight into the ecology of EOs across research areas.

Brousseau (2002) defined three main types of obstacles that students may experience: ontogenetic, didactical, and epistemological. Ontogenetic obstacles occur due to limitations in one's current biological (neuro-physiological) maturation. Didactical obstacles occur when instruction overlooks or potentially exacerbates conceptual challenges in students' mathematical understanding. However, even with instruction designed to target known conceptual challenges, and with sufficient biological maturation (e.g., students who have reached Piaget's formal operational stage), some challenges may persist in students' mathematical reasoning and sense-making. Following Brousseau (2002), we refer to these persistent challenges as *epistemological obstacles* (EOs).

EOs often manifest as a pattern of repeated errors or uncertainties observed by a teacher or researcher. However, EOs are not simply mistakes or the product of ignorance; they are "the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted" (Brousseau, 2002, p. 82). EOs are part of a complex cognitive system and cannot be simply replaced or eradicated (Smith et al., 1994). Our conceptualization of EOs is consistent with what Steffe calls *necessary errors* (Steffe, 1992); they are products of a student's current overall way of operating.

Students may experience an EO when they experience cognitive dissonance, often accompanied by a perturbation, but teachers may experience such dissonance first. Within The Proofs Project, our goal is to identify known EOs in the context of logic and proof, design tasks intended to elicit students' experience of EOs, and leverage inquiry-oriented instruction (Kuster et al., 2018) to provide opportunities for students to address EOs head-on. Consistent with our Piagetian framing, we hypothesize that overcoming an EO requires a structural change in a student's way of operating (i.e., a metamorphic accommodation; Steffe & Thompson, 2000). Understanding connections between EOs can support instructional design for promoting such structural change.

The purpose of this report is to describe a methodology for mapping co-occurrences between EOs as they emerge through mathematical discussion and problem-solving. We first provide an overview of our methodology. We then illustrate this methodology by drawing on preliminary analysis of whole-class data from an introductory proofs course.

Methodology for Mapping Connections Between Epistemological Obstacles

Our method for mapping connections between EOs is summarized in these six steps:

1. Conduct a literature search to identify known EOs related to a particular mathematical idea, generating a codebook consisting of EOs, their definitions, and potential evidence
2. Collect video recordings of students engaging in mathematical discussion or problem solving (with each other and/or with a teacher or researcher)
3. Analyze video recordings using the EO codebook, creating a spreadsheet containing timestamps, key moments in discussion, low-inference notes justifying codes, and important student quotes, adding emergent codes to the codebook that satisfy the definition of EO as needed
4. Decide upon a "chunk" size (e.g., a discussion topic or classroom task) for determining co-occurrence of EOs, and demarcate chunks in the coding spreadsheet
5. Create an adjacency matrix by counting co-occurrences of EOs within each chunk
6. Represent adjacency using a weighted vertex-edge graph (where vertices represent EOs and weighted edges represent number of co-occurrences).

Step 1 is necessary for understanding the EOs that students may face. Being as specific as possible about potential evidence (behavioral indicators) of each EO (as informed by prior research) is critical. To conduct this search, we recommend beginning with a keyword search using Google Scholar, Scopus, and library databases such as Academic Search Complete. Because researchers use disparate terms to refer to persistent challenges in students' mathematical knowing and learning, a broad range of keywords will likely need to be included using AND/OR Boolean operators and asterisks to include characters that may follow a word, such as "misconce* OR challenge* OR error* OR mistake* OR struggle*," along with keywords pertaining to the specific topic area (e.g., "implication OR conditional"). Conducting a systematic review (e.g., Borrego et al., 2014) will yield the largest set of EOs identified in prior literature and potential evidence for each EO.

In Step 2, students are engaged in contexts where EOs may emerge. Often, EOs emerge in the context of discussing or solving mathematical problems or proofs. Step 3 employs standard qualitative analytic techniques, including making inferences about data, categorizing inferences using codes, and writing low-inference notes to justify one's inferences (Maxwell, 2013). Researchers may also identify emergent codes not identified in prior research and apply these to their analysis (e.g., Corbin & Strauss, 2015). Although not reported here, we identified emergent codes from task-based interviews (Clement, 2000) with individual students and applied these to our analysis (see Norton, Arnold, Antonides, & Kokushkin, 2023).

Step 4 is critical and is context-dependent. For us, given our focus on whole-class discussion and the overall instructional design, generating chunks based on general discussion topics and tasks was most practical. However, an analysis of connections between EOs during a clinical interview setting may permit finer-grained chunks. Step 5 involves the creation of the adjacency matrix itself. To ensure consistency, we recommend only counting each distinct EO once within each chunk. For instance, we often coded "Reu" on multiple consecutive lines in our codebook, but if they all occurred within a single chunk, then it would only be counted once. Step 6 allows researchers to visualize connections between EOs and the strength of each connection—especially those greater than 1. Notably, our methodology was inspired by prior research that used adjacency matrices to analyze student reasoning (Selinski et al., 2014). However, our approach is relatively novel within the field, given our theoretical framing and focus on EOs.

Illustrating the Methodology

We now turn to illustrating our proposed methodology by drawing on a preliminary analysis of data from whole-class discussions in an undergraduate introductory proofs course.

Step 1: Literature Review of EOs in the Context of Logic and Proof

Prior research indicates that students often assume that if a logical implication (LI) holds, then its converse holds, treating the LI as biconditional (coded LIff; see Epp, 1999, 2003; Rumain et al., 1983; Wagner-Egger, 2007; Wason & Johnson-Laird, 1972). Such meanings are understandable and common in many non-mathematical contexts (e.g., "If you do your homework, then you can play video games," which is likely intended biconditionally). Transforming LIs into a logically related form, such as its contrapositive or negation, poses additional challenges (coded LIt; see Antonini, 2004; Goetting, 1995; Hawthorne & Rasmussen, 2015; Hub & Dawkins, 2018; Knuth, 2002; Stylianides et al., 2004). In particular, students may reason that the negation of an LI should, itself, be an LI (coded NO; see Arnold et al., in press; Epp, 2003), which we call its "opposite" (e.g., the opposite of $p \rightarrow q$ is $p \rightarrow \sim q$).

Quantifying the negation of a statement poses additional challenges (coded Qneg; see Epp, 2003), particularly uncertainty about whether the negation of an LI should be an existential or universal statement. Many statements in mathematics, including LIs, are not stated with explicit quantification, such as "If a shape is a square, then it is a rectangle." Epp (1999) refers to such statements as containing "hidden" quantifiers (coded Qh), which can pose significant challenges for students. Statements containing multiple quantifiers can pose additional challenges to students (coded Qmult). Students may assume that the order in which two quantifiers are stated does not affect the logical meaning of the statement (coded Qord; for the latter two EOs, see Dawkins & Roh, 2020; Dubinsky & Yiparaki, 2000; Epp, 2003; Piatek-Jimenez, 2010).

Challenges may also emerge when generalizing an argument. To directly prove the validity of a universally-quantified statement, the most common method is to use what Copi (1954) called the Principle of Universal Generalization: a statement holds *for all* $x \in U$ (where U is a universe of discourse) if and only if that statement holds *for any arbitrary* $x \in U$. Students often struggle to make sense of and apply this principle (coded PUG; see Norton, Arnold, Kokushkin, & Tiraphatna, 2023; Selden & Selden, 2013). Additionally, students may attend to the *value* of a variable (e.g., $\delta = .01$), but they may not attend to the *role* played by this variable in a statement or proof, such as the distance between two x-values (coded Qrv; see Dawkins & Roh, 2022).

Lastly, instructors may use visual representations to help students make sense of LIs. Euler diagrams use spatial inclusion and exclusion to illustrate relationships between truth sets of predicate statements. For instance, $p \rightarrow q$ can be interpreted as the truth set of p contained within the truth set of q; using an Euler diagram, this would be (conventionally) represented as one circle (representing the truth set of p) contained within another circle (representing the truth set of q). However, students may struggle to make sense of quantification using Euler diagrams, and students may conceptualize $p \rightarrow q$ as a circle representing the hypothesis *containing* a circle representing the conclusion (coded Reu; see Antonides et al., 2024; Hub & Dawkins, 2018).

Steps 2-4: Data Collection and Analysis

Data come from an upper-level undergraduate introductory proofs course. The instructor (third author) took an inquiry-oriented approach, with students regularly engaging in small-group and whole-class discussions. All class sessions were recorded (excluding exam days). The analysis reported here focuses on discussions from the 40 class sessions throughout the semester. These data were collected from a video camera installed in the back corner of the classroom ceiling, controlled by a researcher in the neighboring room, as well as four microphones installed in the four quadrants of the classroom.

To analyze whole-class data, we used a codebook of EOs developed by (a) searching research literature related to persistent challenges in logic and proof, and (b) supplementing with emergent codes that were identified in clinical interviews with students (Norton et al., accepted). Because emergent codes did not appear in our analysis of whole-class discussion, we focus only on those codes identified in prior literature.

A pair of researchers (first and second authors) created a spreadsheet to record line-by-line descriptions of classroom activity (e.g., tasks, questions from the instructor, and student quotes) with timestamps for each line. They coded each line that corresponded with evidence for one or more EO. Lines were then "chunked" together by task or discussion topic. Two researchers (fourth and fifth authors) created an initial adjacency matrix of EO co-occurrences between chunks, counting each distinct EO once within a given chunk.

Steps 5-6: Results and Interpretation

Table 1 shows an adjacency matrix of EO co-occurrences (omitting EOs that had zero co-occurrences with other EOs). For instance, the first row, second column, shows that LIff and LIt co-occurred 1 time during whole-class discussions. Overall, of the 10 EOs suggested by prior literature, 9 EOs co-occurred with others in at least one segment of class discussion. The final column of Table 1 shows the total number of connections between each EO and other EOs, found by taking row sums. In particular, the total connections for LIt (10) and Qneg (11) indicate that, in the context of class discussions, students' experiences transforming LIs and quantifying the negation of logical statements are highly connected to their experiences of other EOs.

Table 1. Adjacency matrix of EOs experienced in an introductory proofs course

	LIff	LIt	Qh	Qmult	Qord	Qneg	NO	PUG	Reu	Total
LIff		1	1	0	0	0	0	0	3	5
LIt	1		1	0	0	4	1	0	3	10
Qh	1	1		0	0	1	0	1	1	5
Qmult	0	0	0		3	2	1	0	0	6
Qord	0	0	0	3		0	0	0	0	3
Qneg	0	4	1	2	0		3	1	0	11
NO	0	1	0	1	0	3		1	0	6
PUG	0	0	1	0	0	1	1		0	3
Reu	3	3	1	0	0	0	0	0		7

The weighted graph shown in Figure 1 illustrates connections between EOs. Figure 1 makes salient the relatively large number of paths connecting LIff, Reu, LIt, Qneg, and NO. Additionally, Figure 1 shows that Qord is isolated from other EOs except for Qmult.

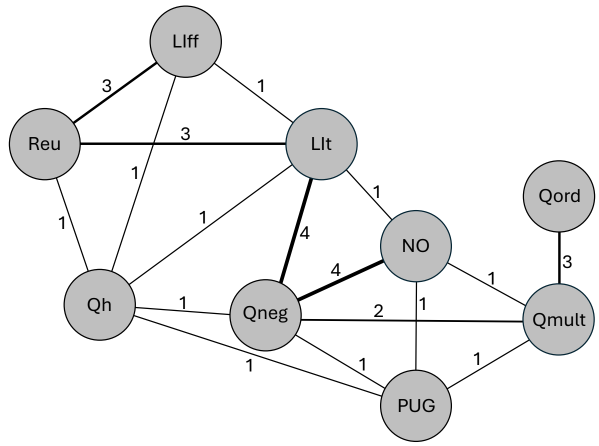

Figure 1. Vertex-edge graph illustrating adjacency among EOs from an introductory proofs course

Classroom Example of a Connection Between EOs

We provide an example of a class discussion intended to elicit students' experience of LIt, NO, and Qneg. The teacher (third author) presented students with the statement p: "For all nonnegative real numbers x, $\sqrt{x} \leq x$" (logically equivalent to the LI: if $x \geq 0$, then $\sqrt{x} \leq x$). The teacher asked students to determine "what statement is equivalent to saying p is false." Following small-group discussion, class discussion began with students mentioning the counterexample $x = ¼$. The teacher then challenged students to generalize this finding into a statement equivalent to saying p is false as follows.

Teacher: ... in general if I say "for all x, something happens," then how would you talk about the other side of the argument?

Student: Um, the false statement would be like, "for all x, something doesn't happen."

Teacher: So, do I need to show that… he said for all x we want to show that open statement doesn't happen.

Student: Oh, for some x in the same set, it doesn't.

In attending to transforming the statement (LIt), the student experienced Qneg and NO; their quantified response "for all x, something doesn't happen" is logically equivalent to the opposite LI, "if x is in the set, then something doesn't happen." The student amended their quantification to "for some x," perhaps in an effort to reconcile their quantification with their understanding that a single counterexample disproves the statement. The adjacency matrix in Table 1 cued us to looking for connections between Qneg and NO, and to proffer explanations for their relatively strong relationship. Indeed, when attempting to transform a universally quantified statement into its negation (LIt), students may initially conceptualize the negation as also needing to be a statement about all values in the universe (Qneg). If this is the case, then it makes sense that students may interpret the negation as its opposite (NO).

Conclusion

This report outlines a methodology for mapping connections between EOs in mathematical discussions and problem solving. Our preliminary analysis of data from an introductory proofs course revealed significant co-occurrences among EOs, such as LIff, Reu, LIt, Qneg, and NO. This methodology can be used broadly to highlight connections between EOs and their potential impact on students' reasoning. Future research should expand this approach across varied mathematical contexts to further refine our understanding of EOs and develop instructional tools and strategies designed to overcome them.

Acknowledgments

This material is based upon work supported by the National Science Foundation under grant number DUE-2141626. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the NSF.

References

Adiredja, A. P. (2020). Students' struggles with temporal order in the limit definition: Uncovering resources using knowledge in pieces. *International Journal of Mathematical Education in Science and Technology*, 1–27. https://doi.org/10.1080/0020739X.2020.1754477

Antonides, J., Norton, A., & Arnold, R. (2024). Linking structures across logic and space: The role of Euler diagrams. *For the Learning of Mathematics*, *44*(2), 2-8.

Antonini, S. (2004). A statement, the contrapositive and the inverse: Intuition and argumentation. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 47–54).

Arnold, R., Kokushkin, V., Norton, A., & Antonides, J. (2024). Students' treatment of the negation of logical implication. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 26th annual conference on research in undergraduate mathematics education* (pp. 294-302). University of Nebraska Omaha.

Borrego, M., Foster, M. J., & Froyd, J. E. (2014). Systematic literature reviews in engineering education and other developing interdisciplinary fields. *Journal of Engineering Education*, *103*(1), 45-76. https://doi.org/10.1002/jee.20038

Brousseau, G. (2002). Theory of didactical situations in mathematics. Kluwer.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Lawrence Erlbaum Associates.

Copi, I. M. (1954). *Symbolic logic*. Macmillan.

Corbin, J., & Strauss, A. (2015). *Basics of qualitative research*. Sage.

Dawkins, P. C., & Roh, K. H. (2020). Assessing the influence of syntax, semantics, and pragmatics in student interpretation of multiply quantified statements in mathematics. International Journal of Research in Undergraduate Mathematics Education, 6(1), 1–22. https://doi.org/10.1007/s40753-019-00097-2

Dawkins, P. C., & Roh, K. H. (2022). Aspects of predication and their influence on reasoning about logic in discrete mathematics. ZDM Mathematics Education, 54(4), 881-893. https://doi.org/10.1007/s11858-022-01332-y

Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA quantification. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education* (Vol. IV, pp. 239–286). American Mathematical Society.

Epp, S. S. (1999). The language of quantification in mathematics instruction. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12* (pp. 188–197). NCTM.

Epp, S. S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly*, *110*(10), 886–899. https://doi.org/10.1080/00029890.2003.11920029

Goetting, M. (1995). *The college students' understanding of mathematical proof* [Unpublished doctoral dissertation]. University of Maryland.

Hawthorne, C., & Rasmussen, C. (2015). A framework for characterizing students' thinking about logical statements and truth tables. *International Journal of Mathematical Education in Science and Technology*, *46*(3), 337–353. https://doi.org/10.1080/0020739X.2014.979895

Hub, A., & Dawkins, P. C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *Journal of Mathematical Behavior*, *50*, 90–102. https://doi.org/10.1016/j.jmathb.2018.02.001
Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, *33*(5), 379–405. https://doi.org/10.2307/4149959
Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2018). Inquiry-oriented instruction: A conceptualization of the instructional principles. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *28*(1), 13-30. https://doi.org/10.1080/10511970.2017.1338807
Maxwell, J. A. (2013). *Qualitative research design: An interactive approach.* Sage.
Norton, A., Antonides, J., Arnold, R., & Kokushkin, V. (accepted). Logical implications as mathematical objects: Characterizing epistemological obstacles experienced in introduction to proofs courses. *Journal of Mathematical Behavior*.
Norton, A., Arnold, R., Antonides, J., & Kokushkin, V. (2023). Epistemological obstacles related to treating logical implications as actions: The case of Mary. In T. Lamberg & D. Moss (Eds.), *Proceedings of the 45th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 71-80). University of Nevada, Reno.
Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2023). Addressing the cognitive gap in mathematical induction. *International Journal of Research in Undergraduate Mathematics Education*, *9*(2), 295–321. https://doi.org/10.1007/s40753-022-00163-2
Piatek-Jimenez, K. (2010). Students' interpretations of mathematical statements involving quantification. *Mathematics Education Research Journal*, *22*(3), 41–56. https://doi.org/10.1007/BF03219777
Rumain, B., Connell, J., & Braine, M. D. S. (1983). Conversational comprehension processes are responsible for reasoning fallacies in children as well as adults: If is not the biconditional. *Developmental Psychology*, *19*(4), 471–481. https://doi.org/10.1037/0012-1649.19.4.471
Selden, A., & Selden, J. (2013). Proof and problem solving at university level. *The Mathematics Enthusiast*, *10*(1/2), 303–334.
Selinski, N. E., Rasmussen, C., Wawro, M., & Zandieh, M. (2014). A method for using adjacency matrices to analyze connections students make within and between concepts: The case of linear algebra. *Journal for Research in Mathematics Education*, *45*(5), 550-583. https://doi.org/10.5951/jresematheduc.45.5.0550
Smith, J. P., diSessa, A. A., & Roschelle, J. (1994). Misconceptions reconceived : A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences*, *3*(2), 115-163. https://doi.org/10.1207/s15327809jls0302_1
Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, *4*(3), 259-309. https://doi.org/10.1016/1041-6080(92)90005-Y
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Lawrence Erlbaum Associates.
Stylianides, A. J., Stylianides, G. J., & Philippou, G. N. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics*, *55*(1–3), 133–162. https://doi.org/10.1023/B:EDUC.0000017671.47700.0b

Wagner-Egger, P. (2007). Conditional reasoning and the Wason selection task: Biconditional interpretation instead of reasoning bias. *Thinking & Reasoning*, *13*(4), 484–505. https://doi.org/10.1080/13546780701415979

Wason, P. C., & Johnson-Laird, P. N. (1972). *Psychology of reasoning: Structure and content*. Harvard University Press.