How do Mathematicians Explain Conditionals?
Diagrams and Explanations for 'if $P(x)$ then $Q(x)$' and '$P(x) \Rightarrow Q(x)$'

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At the introduction to proof, Venn or Euler diagrams are commonly used to illustrate concepts in logic. Standard diagrams show $P(x) \wedge Q(x)$ as the intersection of truth sets for $P(x)$ and $Q(x)$, and $P(x) \vee Q(x)$ as their union. But textbooks rarely include a diagram for the conditional, and those that do use a diagram that differs from the standard form. Does this reflect typical mathematicians' thinking? What do they draw when asked for a diagram to represent the conditional, and does this depend upon sentence formulation? This preliminary report describes a pilot study in which mathematicians and mathematics PhD students were asked to draw a diagram and provide a short accompanying explanation for either 'if $P(x)$ then $Q(x)$' or '$P(x) \Rightarrow Q(x)$'. Results show that they all draw essentially the same diagram, representing a true universal conditional. I illustrate variety in the provided explanations.

Keywords: conditional, diagram, implication, logic

Introduction and Theoretical Background

Venn diagrams are common at the introduction to proof, where they appear in discussions of set theory and logic (Antonides et al., 2024). Diagrams for conjunction and disjunction appear in Figure 1, where the circles represent truth sets for $P(x)$ and $Q(x)$, so the shaded areas contain values of x for which the conjunction $P(x) \wedge Q(x)$ and the disjunction $P(x) \vee Q(x)$ are true.

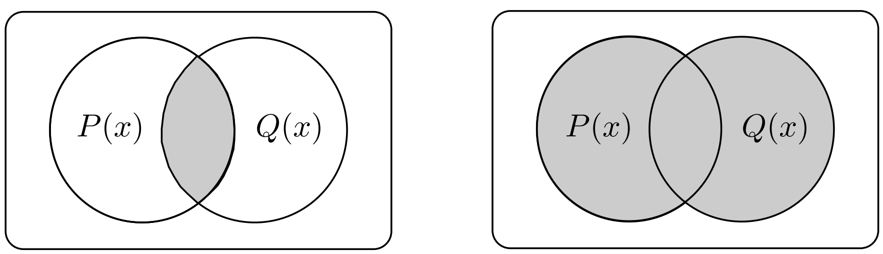


Figure 1. Venn diagrams illustrating conjunction and disjunction.

These diagrams reflect the truth values in Table 1, which additionally shows the truth table of the material conditional 'if $P(x)$ then $Q(x)$', as used in mathematics.

Table 1. Truth tables for conjunction, disjunction, and the material conditional.

$P(x)$	$Q(x)$	$P(x) \wedge Q(x)$	$P(x) \vee Q(x)$	if $P(x)$ then $Q(x)$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

The material conditional is false when $P(x)$ is true and $Q(x)$ is false, and true otherwise, making 'if $P(x)$ then $Q(x)$' equivalent to 'not-$P(x)$ or $Q(x)$'. This leads to a Venn diagram for the conditional as in Figure 2 (Nickerson, 2015) (henceforth, a *truth-value diagram*).

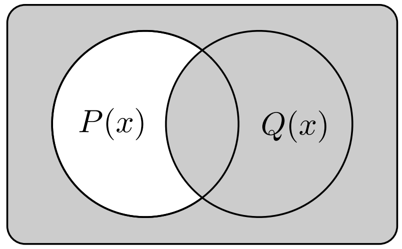


Figure 2. Truth-value Venn diagram for the conditional 'if $P(x)$ then $Q(x)$'.

In everyday language, the material conditional is natural in that we would assert 'if P then Q' when we would assert 'either not P, or P and Q' (Jackson, 1979). But people commonly judge not-P cases as irrelevant to the conditional rather than as cases in which it is true (Newstead et al., 1997). Moreover, they commonly interpret a conditional as asserting an inferential link between its antecedent and consequent, so that 'missing-link' conditionals ('If bats have wings then Paris is in France') sound like nonsense (Bourlier et al., 2023; Douven et al., 2020).

In mathematics, the defined propositional conditional is the material conditional (Durand-Guerrier, 2003). Yet diagrams like that in Figure 2 are rarely seen. Of the 17 introduction-to-proof textbooks most commonly recommended in the UK and the US (Alcock & Sa, in press), only two include a diagram in their explanations of conditionals, and neither use a truth-value diagram. Instead, both give Euler diagrams in which the truth set of $P(x)$ is contained in the truth set of $Q(x)$, as in Figure 3 (henceforth, *inclusion diagrams*).

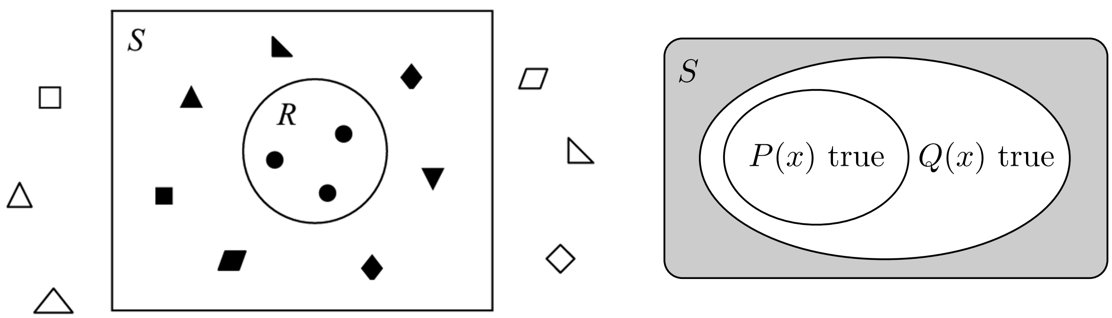


Figure 3. Diagrams for conditionals, formulated as R ⇒ S (round ⇒ solid) in Lay (2014, p.5) and as $P(x) \Rightarrow Q(x)$ in Stewart & Tall (2015, p.134).

In inclusion diagrams, the idea of representing all possible combinations of truth values for $P(x)$ and $Q(x)$ is abandoned in favour of a diagram in which the conditional 'if $P(x)$ then $Q(x)$' is universally true (there are no x values for which $P(x)$ is true and $Q(x)$ is false). The reason for this is suggested in the explanation offered by Stewart and Tall:

> "We are really interested only in the case where $P(x) \Rightarrow Q(x)$ is true for all x. In this case, if $P(x)$ is true, so must $Q(x)$ be; that is, if $a \in \{x \in S | P(x)\}$ then $a \in \{x \in S | Q(x)\}$ which means $\{x \in S | P(x)\} \subseteq \{x \in S | Q(x)\}$." (Stewart & Tall, 2015, p.134)

The claim that mathematicians are primarily interested in true universal conditionals is arguably reflected in other textbooks' use of 'P implies Q' or '$P \Rightarrow Q$' as straightforward alternatives to 'if P then Q' (Alcock & Sa, in press). The language of implication puts mathematicians at odds with philosophers, who usually denote the material conditional '$P \supset Q$' ('P hook Q'; Edgington, 2020) and are wary of ambiguity around '⇒' and 'implies', reserving these for argument forms or avoiding them altogether (Smith, 2020). But the mathematical use corresponds comfortably to natural language for true universal conditionals: where 'for all x, if $P(x)$ then $Q(x)$' is true, $Q(x)$ can be deduced from $P(x)$ (e.g., Epp, 2004).

Of course, it could be that the diagrams in Figure 3 are idiosyncratic: perhaps mathematicians do not typically represent these constructs in this way. Or it could be that formulation matters: perhaps '$P(x) \Rightarrow Q(x)$' invites an inclusion diagram but 'if $P(x)$ then $Q(x)$' would be more likely to elicit a truth-value diagram. I therefore intend to investigate diagrams produced by a wider sample of mathematicians, along with explanations they provide for those diagrams, and features that they and students value in diagrams and explanations produced by others. This preliminary report describes a pilot study addressing three research questions:

1. What Venn/Euler diagrams do mathematicians produce for the conditional?
2. Does this vary between the formulations 'if $P(x)$ then $Q(x)$' and '$P(x) \Rightarrow Q(x)$'?
3. How do mathematicians explain their diagram(s)?

Method

Mathematicians were recruited at the 2024 British Mathematical Colloquium, where attendees are mathematicians and PhD students working largely in pure mathematics. They participated voluntarily in response to invitations in an email and a talk. Each participant read an information sheet and signed a consent form that asked for their main area of mathematical work (pure, applied, statistics/data science, other) and their experience level (PhD student, postdoc, lecturer[1], other). Fourteen people participated, of whom:

- 9 were PhD students in pure mathematics (PhD P in Table 2);
- 1 was a PhD student in applied mathematics (PhD A);
- 1 was a postdoc in pure mathematics (PD P);
- 3 were lecturers in pure mathematics (Lec P).

Participants responded (under no time constraint) to the prompt:

> Please draw a Venn/Euler diagram to represent 'if $P(x)$ then $Q(x)$' ['$P(x) \Rightarrow Q(x)$'].
> Then add a short explanation suitable for undergraduate students.

Half received the 'if $P(x)$ then $Q(x)$' prompt and half the '$P(x) \Rightarrow Q(x)$' prompt; all had most of a page in which to respond. Their responses were analysed qualitatively as below.

Analysis and Results

To address Research Question 1, I first analysed the diagrams. This was a quick process because all participants drew inclusion diagrams. Two then added a truth-value diagram, and one of these offered two further alternative diagrams; these I will present at the conference. The inclusion diagrams varied in three respects:

1. Some labelled the sets $P(x)$ and $Q(x)$ and some just used P and Q;
2. Some framed the sets within a universal set and some did not;
3. Some included extra labelling or shading.

Table 2 summarises the features of each inclusion diagram and shows which participants included a truth-value diagram. It also shows that the answer to Research Question 2 appears to be that formulation of the conditional does not affect the Venn/Euler diagrams that mathematicians draw, at least not in the way anticipated. Participants responded with the same (first) diagram regardless of the prompt, and the two truth-value diagrams appeared in response to the formulation '$P(x) \Rightarrow Q(x)$', not 'if $P(x)$ then $Q(x)$'. Of course, this is a small sample.

[1] In the UK, the term 'lecturer' is equivalent to the North American 'professor', and people specialise early so that a PhD is a research-only degree.

Table 2. Features of inclusion diagrams: TV = provided a second, truth-value diagram.

Prompt	Participant	Labels	Univ	Extra labels/shading	TV	Explanation
if-then	PhD P	$P(x)$				Basic
if-then	PhD P	$P(x)$				Converse
if-then	PhD P	$P(x)$		P shaded		Background
if-then	PhD P	P	Yes			Converse
if-then	PhD P	P	Yes	specific points		Converse
if-then	PD P	P				Basic
if-then	Lec P	$P(x)$				Background
⇒	PhD P	$P(x)$				Converse
⇒	PhD P	P				Converse
⇒	PhD P	$P(x)$		'x for which $P(x)$ holds'	Yes	Basic
⇒	PhD P	P	Yes			Basic
⇒	PhD A	$P(x)$		'this area is a subset of this area'		Background
⇒	Lec P	$P(x)$		'points here satisfy $P(x)$ and $Q(x)$'		Converse
⇒	Lec P	$P(x)$		$Q(x)$ shaded, $P(x)$ cross-hatched	Yes	Basic

To address Research Question 3, I analysed the particpants' explanations for the inclusion diagrams (Table 2, last column). These varied primarily in whether and how they went beyond a basic explanation of the diagram's structure. Some gave just that basic explanation, as in Figure 4. Some additionally noted that the converse of the conditional might not be true, as in Figure 5. Some provided comments on the background assumed when mathematicians read and write conditionals (cf. Dawkins & Roh, 2024), as in Figure 6.

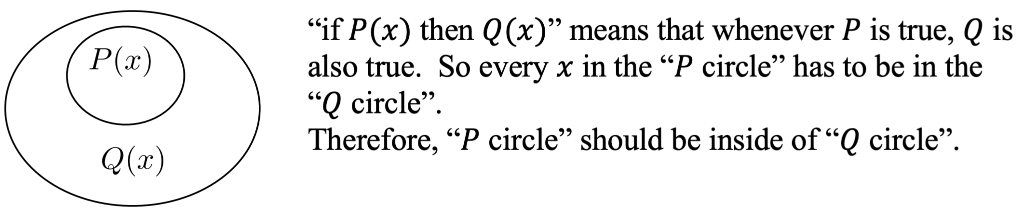


Figure 4. Diagram with basic explanation.

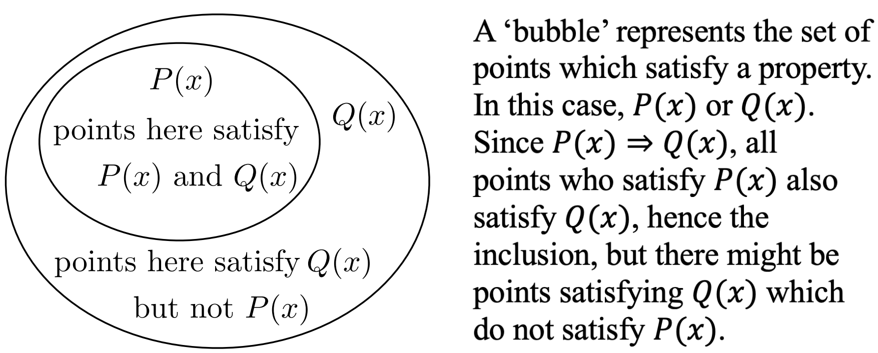
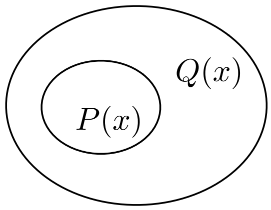


Figure 5. Diagram with significance of area outside inner set.



Note that "$P(x) \Rightarrow Q(x)$" is not a statement (or sentence) as it contains a free variable. So, given a suitable structure and a point, say α, in that structure we can test whether the premise $P(\alpha) \Rightarrow Q(\alpha)$ holds in the structure or not. Namely, whether property P holding at α "implies" that Q holds at α.
Also note that by adding a quantifier we can make this into a statement; for example: "$\exists x\, P(x) \Rightarrow Q(x)$" or "$\forall x\, P(x) \Rightarrow Q(x)$". Normally, or most commonly, we look at the latter, that is
"$\forall x\, P(x) \Rightarrow Q(x)$".
This, in a given structure, is to be interpreted as for all elements property P implies property Q. In other words, the elements that satisfy property P form a subset of those elements satisfying Q.

Figure 6. Diagram with explanation including assumed background.

Discussion

This pilot study investigated diagrams mathematicians draw for the conditional, whether these diagram are affected by formulation of the conditional, and what explanations mathematicians offer for their diagrams. Regardless of formulation, participants all drew inclusion diagrams first, with a small minority adding truth-value diagrams. Both diagrams and explanations reflected those offered in Stewart and Tall's (2015) book in that mathematicians appear to think primarily about true universal conditionals; some additionally explained non-equivalence of the converse and some added material on background assumptions about propositions and quantification.

I plan to extend this work in two directions.

First, I will conduct a study in which a larger number of mathematicians respond to the same prompts, except with a note that they can, if they wish, include more than one diagram. I expect this to confirm that inclusion diagrams are favoured. I will then ask both mathematicians and undergraduates to use comparative judgement to rate the quality of (a selection of) the resulting explanations, coupling the results with a content analysis to identify and compare features of diagrams and explanations that are valued by the two groups (cf. Davies, Alcock & Jones, 2021).

Second, I will ask mathematics students in introduction-to-proof courses to select which of a range of diagrams best represent '$P(x) \wedge Q(x)$', '$P(x) \vee Q(x)$' and 'if $P(x)$ then $Q(x)$'. Asking about all three connectives will permit more sensible inclusion of a wider range of diagrams and might reveal non-standard interpretations. For '$P(x) \wedge Q(x)$', one can imagine that students might select the diagram on the right of Figure 1. For 'if $P(x)$ then $Q(x)$', they might select an inclusion diagram with $Q(x) \subseteq P(x)$ – this would be natural if they imagine restricting attention to P cases and asking whether these must also be Q cases, as in the Ramsey test (Ramsey, 1929) (one participant in this pilot study drew such diagram then crossed it out). This work might also reveal consistency or otherwise in how students interpret shading in relation to truth sets.

I will report progress of research in both directions at the conference.

Acknowledgments

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References

Alcock, L. & Sa, R. (in press). How do introduction-to-proof textbooks explain conditionals and implications? In press at *International Journal of Research in Undergraduate Mathematics Education*.

Antonides, J., Norton, A., & Arnold, R. (2024). Linking structures across logic and space: The role of Euler diagrams. *For the Learning of Mathematics*, *44*, 2–8.

Bourlier, M., Jacquet, B., Lassiter, D., & Baratgin, J. (2023). Coherence, not conditional meaning, accounts for the relevance effect. *Frontiers in Psychology*, *14*, 1150550.

Davies, B., Alcock, L., & Jones, I. (2021). What do mathematicians mean by proof? A comparative-judgement study of students' and mathematicians' views. *Journal of Mathematical Behavior*, *61*, 100824.

Dawkins, P. C. & Roh, K. H. (2024). Unitizing predicates and reasoning about the logic of proofs. *Journal for Research in Mathematics Education*, *55*, 76–95.

Douven, I., Elqayam, S., Singmann, H., & van Wijnbergen-Huitink, J. (2020). Conditionals and inferential connections: Toward a new semantics. *Thinking & Reasoning*, *26*, 311–351.

Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics*, *53*, 5–34.

Edgington, D. (2020). Indicative conditionals. In E.N. Zalta, (Ed.), *The Stanford Encyclopedia of Philosophy (Fall 2020 Edition)*. https://plato.stanford.edu/archives/fall2020/entries/conditionals/.

Epp, S. S. (2004). Discrete mathematics with applications (3rd Edition). Thomson Brooks/Cole.

Jackson, F. (1979). On assertion and indicative conditionals. *Philosophical Review*, *88*, 565–589.

Lay, S. R. (2014). *Analysis with an introduction to proof (5th edition)*. Pearson.

Newstead, S. E., Ellis, M.C., Evans, J. St. B. T., & Dennis, I. (1997). Conditional reasoning with realistic material. *Thinking & Reasoning*, *3*, 49–76.

Nickerson, R. S. (2015). *Conditional reasoning: The unruly syntactics, semantics, thematics, and pragmatics of "if"*. Oxford University Press.

Ramsey, F. P. (1929). General propositions and causality. In D.H. Mellor (Ed.), *Foundations: Essays in philosophy, logic, mathematics and economics*, pp.145–163. Humanities Press.

Smith, P. (2020). *An introduction to formal logic: Second Edition*. Cambridge University Press.

Stewart, I. & Tall, D. (2015). *The Foundations of Mathematics (2nd Edition)*. Oxford University Press.