How do Introduction-to-Proof Textbooks Explain Conditionals? Mathematical, Natural-Language and Hybrid Examples

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Mathematics defines the propositional conditional 'if P then Q' as the truth-functional material conditional: it is false if P is true and Q is false, and true otherwise. This is counterintuitive with respect to natural language in that 'missing-link' conditionals (with true but unconnected P and Q) are considered true, as are conditionals with false antecedents. How is this reflected in the examples used by introduction-to-proof textbooks? Do these use natural-language conditionals in their explanations, or avoid them in favour of mathematical conditionals? What characteristics do their chosen examples share, and where do they diverge? We address these questions using a theoretically informed qualitative analysis of content on conditionals in 17 commonly recommended introduction-to-proof textbooks. We consider advantages and disadvantages of different approaches in relation to research in philosophy and psychology as well as in undergraduate mathematics education.

Keywords: conditional, example, implication, logic, reasoning.

Introduction

Undergraduate mathematics students encounter numerous mathematical conditionals, in theorem statements and in more informal claims and conjectures. In content courses, most of these are *universal conditionals* (also called *generalized conditionals*, Durand-Guerrier, 2003) of the form 'if $P(x)$ then $Q(x)$', interpreted to mean '$\forall x$, if $P(x)$ then $Q(x)$'. It is important for students to distinguish such conditionals from their converses and inverses, so that they can avoid errors in constructing and validating proofs (e.g. Selden & Selden, 2003).

In introduction-to-proof courses, students might also encounter *propositional conditionals* of the form 'if $P(x_0)$ then $Q(x_0)$', where x_0 is a specific object. They learn that these are treated according to a *material* interpretation, so that 'if $P(x_0)$ then $Q(x_0)$' is false if $P(x_0)$ is true and $Q(x_0)$ is false, and true otherwise. This gives the truth conditions that we want because it means that a universal conditional is refuted only by examples that we understand as counterexamples. However, it is peculiar in relation to natural language. In everyday settings, people tend to think that 'missing-link' conditionals – e.g. 'if bats have wings then Paris is in France' (Bourlier et al., 2023) – are false or nonsensical, not true. They also usually think that for cases where $P(x_0)$ is false, 'if $P(x)$ then $Q(x)$' is irrelevant rather than true (Newstead et al., 1997).

How is this reflected in the examples used by introduction-to-proof textbooks to explain conditionals? Do texts use natural-language conditionals to support understanding of the material conditional? Or do they avoid natural language, giving up possible analogies in favour of more clear-cut mathematical examples? Do they include missing-link or false-antecedent conditionals? To address these questions, we analysed examples used to explain conditionals in commonly recommended introduction-to-proof textbooks. We consider advantages and disadvantages of different approaches in relation to research in philosophy and psychology.

Theoretical Background

The mathematical conditional has a stipulated meaning: the propositional connective 'if-then' conforms to the standard truth table for the material conditional (Table 1, with illustrative content). This connective is truth-functional, meaning that the truth value of ‘if $P(x_0)$ then $Q(x_0)$’ depends only upon the truth values of $P(x_0)$ and $Q(x_0)$. As noted above, it underpins the mathematically desirable interpretation for universal conditionals: the universal conditional ‘if $P(x)$ then $Q(x)$’ is not falsified by cases where $P(x_0)$ is false or $Q(x_0)$ is true (Edgington, 2020). For instance, the conditional ‘if x is a circle then x is an ellipse’ is universally true – it is not falsified by circles, non-circular ellipses, or squares (see Table 1).

Table 1. Truth table for the material conditional with ‘if x is a circle then x is an ellipse’.

	circle	ellipse	if circle then ellipse
circle	T	T	T
(impossible)	T	F	F
non-circular ellipse	F	T	T
square	F	F	T

However, adopting the material interpretation requires sacrifices because we usually subject natural-language conditionals to an *inferential* reading, assuming that the consequent is caused by the antecedent or otherwise follows reliably from it (Douven et al., 2020). This makes missing-link conditionals sound false or nonsensical. Also, we are usually uncomfortable with with false-antecedent cases, considering conditionals irrelevant in these (Newstead et al., 1997). The picture is further complicated by the fact that natural language often allows ‘if’ to be interpreted as ‘if and only if’. This is fairly standard in conditional promises, threats, and statements about permission (Nickerson, 2015): from ‘if you mow the lawn, then I will give you \$5’, a person could reasonablly to infer that if they do not mow the lawn, they will not receive \$5 (Cummins et al., 1991).

In mathematics, we carefully distinguish ‘if’ from ‘if and only if’. But otherwise the pragmatics is similar. In theorems and other claims, we usually assert ‘if $P(x)$ then $Q(x)$’ or ‘$P(x) \Rightarrow Q(x)$’ only when proposing that for all relevant *x*, $Q(x)$ follows deductively from $P(x)$. Thus, we would happily say that ‘if *n* is divisible by 6, then *n* is divisible by 3’ is true and ‘if *n* is odd, then *n* is prime’ is false. However, we would not usually consider the corresponding propositional conditionals individually. These sound natural when a universal conditional is true and the truth value of the antecedent of a corresponding propositional conditional is unknown, as in ‘if 4,686 is divisible by 6 then 4,686 is divisible by 3’ (Epp, 2004, p.17). But the natural-language use of conditionals to indicate uncertainty (Grice, 1989) means that they sound unnatural where the truth values are known:

- if 12 is divisible by 6 then 12 is divisible by 3 (true-true case so true);
- if 9 is divisible by 6 then 9 is divisible by 3 (false-true case so true);
- if 8 is divisible by 6 then 8 is divisible by 3’ (false-false case so true).

They sound even more unnatural when a universal conditional is false:

- if 3 is odd then 3 is prime (true-true case so true);
- if 9 is odd then 9 is prime (true-false case so false);
- if 2 is odd then 2 is prime (false-true case so true);
- if 8 is odd then 8 is prime (false-false case so true).

These sound strange because we cannot infer primeness from oddness; the ‘2’ and ‘8’ cases sound additionally awkward due to their false antecedents.

Introduction-to-proof texts therefore have issues to consider in the examples that they use. They could engage with the peculiarities of the material conditional, highlighting missing-link and false-antecedent cases. Or they could focus primarily on conditionals that are more ‘natural’ in everyday language and in mathematics in that they are universal and submit to inferential interpretation. These choices will have consequences: students’ understandings of mathematical concepts are heavily influenced by their experience with examples (Sinclair et al., 2011). Even if a text defines the material conditional clearly, the examples it uses for illustration will influence students’ conceptual understanding. In the study reported here, we investigated what introduction-to-proof texts do by asking, for their example conditionals:

1. What balance is struck between mathematics and natural language?
2. What proportions have inferential versus missing links?
3. What other features are evident (propositional vs. universal, specifc content, etc.)?

Method

We assembled a list of introduction-to-proof texts commonly recommended in the US and the UK. The US list was compiled by David & Zazkis (2019); it contained one book per course. The UK list we compiled by identifying introduction-to-proof courses and using websites or contacting lecturers to acquire book lists. Of 66 courses identifiable via the UCAS website of UK university degree programmes, we acquired lists for 52 (nine did not have a list, and five did not respond to enquiries or follow-ups). The US and UK lists were not equivalent because UK courses do not usually have a single ‘course book’ or identify a main book on their list. We do not consider this a methodological problem because we were not trying to conduct a representative survey; we simply wished to select texts that students were likely to consult.

From the two lists, we identified all books recommended at five or more institutions across the two countries. One of these had no specific content on conditionals, one had minimal content, and four discussed only universal conditionals, usually with a focus on distinguishing a conditional from its converse (Alcock & Sa, in press). We do not discuss these further. The 12 texts that discussed both universal and propositional conditionals are listed in Table 2, together with recommendation numbers in the US and the UK.

Table 2. Texts Analysed, with Numbers of Recommendations.

Author(s)	Title	UK	US	Total
Houston	How to think Like a Mathematician	25	2	27
Chartrand, Polimeni, Zhang	Mathematical Proofs: A Transition to Advanced Mathematics		17	17
Smith, Eggen, St Andre	A Transition to Advanced Mathematics		15	15
Velleman	How to Prove it: A Structured Approach	6	7	13
Eccles	An Introduction to Mathematical Reasoning	6	4	10
Rosen	Discrete Mathematics and Its Applications	3	7	10
Hammack	Book of Proof	3	6	9
Epp	Discrete Mathematics with Applications	1	8	9
Stewart & Tall	Foundations	6		6
Lay	Analysis with an Introduction to Proof	1	4	5
D’Angelo, West	Mathematical Thinking: Problem-Solving and Proofs		5	5
Sundstrom	Mathematical Reasoning: Writing and Proof		5	5

For each text, we listed all example conditionals in the section(s) where conditionals were explained, regardless of whether these were expressed using ‘if-then’ or ‘implies’. We also included biconditionals and conditionals deliberately formulated differently, e.g., ‘You can run for president only if you are a citizen’ (Velleman, 2006, p.49). Counting was not straightforward because some texts used the same conditional multiple times to make or clarify different points. Because we were interested in the content used, we decided that a single conditional used multiple times would count once. Across the 12 books, we identified 162 conditionals in total.

Analysis and Results

To address Research Question 1, we classified each conditional as *mathematical* or *natural-language*; we also introduced a third category, *hybrid*, because some texts included conditionals involving content of both types. Illustrations appear below.

- *Mathematical:* If x is even then x^2 is even (Houston, 2009, p.64).
- *Natural:* If you pass the final exam, then you will pass the course (Hammack, 2013, p.43).
- *Hybrid:* $\pi = 3 \Rightarrow$ Paris is the capital of France (Smith et al., 1997, p.11).

This classification revealed considerable differences in approach, as documented in Table 3. Some texts used very few conditionals, sticking to mathematical content; some used mainly natural-language conditionals; some used lots of each. Only three used hybrid conditionals.

Table 3. Counts of mathematical, natural-language and hybrid conditionals in the analysed texts, with sub-classification as having inferential or missing links.

	Houston	Chartrand et al.	Smith et al.	Velleman	Eccles	Rosen	Hammack	Epp	Stewart & Tall	Lay	D'Angelo & West	Sunstrom	Total
Mathematical													
Inferential	21	3	13	1	3		7	7	1	7	2	7	72 (45%)
Missing	1	5	7		2					1			16 (10%)
Natural													
Inferential	12	1	2	16	1	8	2	16		3		4	65 (41%)
Missing						1		1					2 (1%)
Hybrid													
Missing			3			1		1					5 (3%)

To address Research Question 2, we classified each conditional as having an *inferential* or *missing* link. All hybrid conditionals had missing links – these were used to make the point that the truth value of a propositional conditional does not require a link between its antecedent and consequent. Some texts made the point explicitly; others relied on readers to notice what they were supposed to learn. Illustrative mathematical and natural-langauge conditionals with missing links appear below (these contrast with the inferential links above).

- *Mathematical:* $(\pi < 3) \Rightarrow (1 + 1 = 2)$ (Eccles, 2007, p.12).

- *Natural:* If computers are machines, then Babe Ruth was a baseball player (Epp, 2004, p.26).

Table 3 shows that the texts' example conditionals overwhelmingly had inferential links[1]. Five texts included none with missing links, and those that did use them included only small numbers. This presumably reflects the fact that while introduction-to-proof courses commonly cover the material propositional conditional, mathematicians are primarily interested in developing students' understanding of universal conditionals.

To address Research Question 3, we sliced the data differently, classifying the conditionals as *propositional* or *universal* with additional (sub)classifications to distinguish features relevant to the theoretical background. Four conditionals were peculiar in some way and did not fit our classification scheme[2]. The remaining 158 did, and we discuss each classification in turn.

Twenty condtionals (~8%) were unambiguously *propositional*. Of these, four were hybrid and had missing links (like the example above). Two were natural-language; these had a missing and an inferential link respectively (again, see above). The remaining 14 were mathematical; nine had missing links and four had inferential links either with a false antecedent or in a false claim. One (from Epp, repeated below) had an inferential link but uncertain truth values of the antecedent and consequent, so worked particularly well with both the logic of the material conditional and the pragmatics of natural language.

- If 57 is prime, then 3 is odd (Chartrand et al., 2018, p.20).
- $-1 = 1 \Rightarrow 1^2 = 1$ (Houston, 2009, p.65).
- If 4,686 is divisible by 6, then 4,686 is divisible by 3 (Epp, 2004, p.17).

Thirty-four conditionals (~22%) had *ambiguous* status in relation to the propositional/ universal distinction because they used natural language (one was hybrid) and invoked a named person (sometimes 'I') or a physical situation. Six of these were promises and two were threats; the remainder were more general assertions. Asserted with clear specific referents, these would be propositional. However, all had inferential links between antecedent and consequent, and readers were usually asked to consider how they would apply in multiple situations, so they were effectively treated as more general claims. Some would naturally be interpreted in line with the logic of the material conditional, others less so; as noted above, everyday promises and threats would often be interpreted as biconditionals.

- If it stops raining by Saturday, then I will go to the football game (Lay, 2014, p.4).
- If you do not get to work on time, then you are fired (Epp, 2004, p.20).
- I am not American implies that I am not Texan (Houston, 2009, p.73).
- If John didn't go to the store, then we're out of eggs (Velleman, 2006, p.44).

The remaining 104 conditionals were unambiguously *universal*.

Of the mathematical universal conditionals, 50 (~33% of the total) were *universal assertions*. Two involved more than one variable; we classified these as having missing links but note that a link is possible with additional specification. The remaining 48 were mathematical claims that could be theorems or conjectures; two were clearly intended to have uncertain truth value, nine were clearly false, the remaining 36 were clearly true. Readers were often invited to consider these in relation to specific cases and thereby to consider propositional conditionals, but their statements were universal.

[1] Two conditionals did not fit this classification: Lay's (2014, p.7) 'If roses are red, and violets are blue, then I love you' and D'Angelo & West's (1999, p.32) 'If S is "This book was published in the year 73', then $S \Rightarrow P$ is true'.

[2] Those in footnote 1, plus Smith, Eggen & St Andre's (1997, p.11) 'If $4n > 10$ then 1 is prime' and Rosen's (2007, p.7) 'If today is Friday, then $2 + 3 = 5$'.

- $f(a) = 0 \Rightarrow a > 0$ (Eccles, 2007, p.14).
- If $x^2 > 5$, then $x^3 + 12x + 4 < 3$ (Houston, 2009, p.64).
- If p is a prime number, then p is odd (Hammack, 2013, p.57).
- $x > 0 \Rightarrow x + 1 > 1$ (D'Angelo & West, 1999, p.32).

A further seven mathematical universal conditionals (~4%) referred in their antecedents to *specific objects*. These are interesting from a theoretical perspective because such statements are common in mathematics and are technically universal but represent a sort of extreme case – universal statements 'normally' allow multiple different cases for a true antecedent. For some, a student's prior experience might lead to natural consideration of false antecedent cases; for others, it is harder to imagine that it would.

- If $x = 3$ then $x^2 = 9$ (Sundstrom, 2016, p.58).
- If $x = 2$, then x is a solution to $x^2 = 2x$ (Smith et al., 1997, p.14).

Finally, ten mathematical universal conditionals (~6%) captured *universal definitional* relationships, either standard definitions or local definitions constructed for a specific task. A further six (~4%) were definition-like in using categorical *inclusion* relationships. We note that standard definitions are biconditionals, but this and their status as definitions was only sometimes highlighted. Categorical inclusions clearly conform to the logic of the material conditional (in a sense, all mathematical conditionals capture categorical inclusions, but this was more explicit here).

- $a < 0 \Rightarrow |a| = -a$ (Smith et al., 1997, p.14).
- If $g(m, n) = n^2 + n + m$, then $g(n, n + 1) = (n + 1)^2$ (Lay, 2014, p.22).
- If a real number is an integer, then it is a rational number (Epp, 2004, p.82).

Of the natural-language universal conditionals, five (~3%) captured *universal inclusions* due to category membership or geographic location. These correspond clearly to the logic of the material conditional, so it is perhaps surprising that they were not more often used. Eleven (~7%) we classed as *universal associations* because they had inferential connections, but these arose from human conventions or plans or observations rather than category inclusions or physical causes. Fifteen (~9%) were *universal permissions, promises or threats*. As noted before, these would not always be interpreted according to the logic of the material conditional. Often, they would support pragmatically valid *denial of the antecedent* inferences: for the last two below, the hearer might reasonably infer that if there are not ten people, the lecture will not be given, and that if they do not clean their room, they will not be allowed to watch TV. Such pragmatic issues were sometimes but not always discussed.

- Being an article of clothing is necessary for being a glove (Houston, 2009, p.72).
- If x is a student in this class, then x has visited Mexico or Canada (Rosen, 2007, p.43).
- You can work here only if you have a college degree (Lay, 2014, p.5).
- If at least ten people are there, then the lecture will be given (Velleman, 2006, p.49).
- If you do not clean your room, then you cannot watch TV (Sundstrom, 2016, p.57).

Discussion

This work addressed three research questions about examples used in introduction-to-proof texts' discussion of conditionals:

1. What balance is struck between mathematics and natural language?
2. What proportions have inferential versus missing links?
3. What other features are evident (propositional vs. universal, specifc content, etc.)?

In answer to Question 1, the balance of mathematical versus natural-language conditionals varied dramatically by text: authors do not agree on how to use different content. In answer to Question 2, a large majority of conditionals had inferential links. In line with our theoretical framing, we suggest that this reflects a mathematical interest in true universal conditionals: these can be treated inferentially, and mathematicians are interested in the corresponding proofs. In answer to Question 3, only about 8% of the conditionals used were unambiguously propositional; a further 22% could potentially be propositional but were arguably treated as universal claims. Of the unambiguously universal conditionals, a large proportion (33% of the total) were mathematical claims that could form theorems or conjectures. Smaller proportions of the mathematical conditionals had specific-object antecedents or involved defining or categorical inclusion relationships. Universal natural-language conditionals were spread across inclusions, associations, permissions, promises and threats. Of these, the inclusion relationships most naturally conform to the logic of the material conditional.

This analysis reveals the complexity of choices facing textbook authors and instructors. What influence might their choices have on student learning?

Using conditionals with inferential links has advantages and disadvantages. Students will likely find these 'natural', but they obscure the fact that the truth value of the material conditional depends only upon the truth values of its antecedent and consequent. So their likely utility depends upon the goal. If we want to teach students how to to interpret the conditionals that they will encounter in content courses in undergraduate mathematics, then the collective emphasis is probably reasonable. At least, it is reasonable if we assume that students will overlook the peculiarities of the material conditional. For students who notice those, explanations are somewhat thin on the ground (see also Alcock & Sa, in press).

If the aim is to teach the material conditional as part of mathematical logic, then the examples offered provide only partial exposure to relevant issues. Only some texts confronted the peculiarities head on, explicitly writing out propositional conditionals in order to reflect upon how they relate to the material conditional interpretation. Most avoided this by presenting universal conditionals and inviting thought about how they apply to particular cases. As we know that people often consider conditionals nonsensical when they have missing links (Bourlier et al., 2023) and irrelevant in false-antecedent cases (Newstead et al., 1997), more substantive discussion would likely be needed in order to help students connect their understanding of familiar mathematics or everyday situations to formal logic.

We conclude with two issues worthy of note. First, those conditionals that most naturally conform to a material interpretation were mathematical and natural-language conditionals capturing inclusions in the form of categorical or geographical relationships. Permissions, promises and threats are riskier as these are often interpreted as biconditionals; this does not mean that we should avoid them, but we suggest that they might need more in-depth discussion. Second, we saw only one use of a simple and clearly true inferential universal conditional applied in cases where the truth value of the antecedent would be unknown (Epp, 2004, p.17, 'if 4,686 is divisible by 6, then 4,686 is divisible by 3'). This seems particulary useful in linking mathematical logic with everyday pragmatics, and we suggest that the educational effects of exposure to this and other types of example merit theoretical consideration and empirical testing.

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