Instructors' Grounding Metaphors for Quotient Groups

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In this report, we examine the use of grounding metaphors across three abstract algebra instructors in their discussion of quotient group instruction. We identified three primary categories: construction metaphors, equivalence metaphors, and positional metaphors. The construct metaphors were further refined into building metaphors and demolition metaphors. In this paper, we provide an overview of the types of grounding metaphors in usage and provide insight into how the different instructors took on these metaphors when describing quotient groups and their instruction. We conclude with some considerations as to how these metaphors provide insight into quotient structure and considerations for future research.

Keywords: Metaphorical Thinking, Abstract Algebra, Instruction, Quotient Groups

Quotient groups have been identified as one of the most important, but also one of the most difficult topics in Abstract Algebra (Melhuish, 2019). Most of the research conducted regarding quotient groups has focused on how students think about quotient groups (e.g., Asiala et al., 1997; Melhuish et al., 2023). This literature base suggests students might develop only partial understandings such as relying on coset generation procedures (Hazzan, 1999) or seeing elements of quotient groups as sets and elements but not both (Asiala et al., 1997; Siebert & Williams, 2003). Some researchers have suggested that meanings related to partitioning are more productive as they are compatible with viewing quotient groups as the preimage of a homomorphism (Melhuish et al., 2023) and can support a richer understanding of why only certain subgroups can form quotient groups (Larsen & Lockwood, 2013). This work suggests that partitioning might serve as one important grounding metaphor for quotient groups. However, there is significantly less known about the way in which instructors conceptualize quotient group, what other metaphors may exist, and how they are taught. The goal of this proposal is to contribute to the literature on the teaching of quotient groups by addressing the use of metaphors in instruction. The research questions being addressed is:

What grounding metaphors about quotient groups are used during abstract algebra instructors' descriptions of their teaching?

Literature Review

There is a great deal of consistency in the teaching of proof-based courses with the genre of chalk-talk dominating the way in which most of the courses are taught (Melhuish et al., 2022). That is, instructors lecture, writing formal mathematics on the boards, verbalize what is written and provide additional, often informal, insights, and ask students some form of questions (e.g., Artemeva & Fox, 2011; Paoletti et al., 2018). This consistency may mask important variations in instruction. For example, Pinto (2019) compared two mathematicians teaching a real analysis course using the same lesson plans. Despite this common material, they differed in the general enactment of the lesson including the timing of when parts were enacted and relevant to this paper, with their use of metaphorical imagery.

Pinto (2019) discussed how one of the mathematicians, Amit, used metaphorical language and imagery to unpack the definition of the derivative. It was stated that Amit wanted to help the students “obtain a picture of it” (p. 10). Thus, there was use of visual images and colloquial language including phrases such as “line”, “swing,” and “trap.” This contrasted with Yoav, who, rather than using imagery-based language approached the definition of a derivative term by term.

Rupnow (2021) similarly found a range of conceptual metaphors used by the mathematicians in their discussion of homomorphisms and isomorphisms. These metaphors spanned ideas of sameness and mappings and specific metaphors used by the mathematicians varied both in when and how often they occurred. Olsen et al. (2020) identified that instructors also use a number of metaphors conveying ideas of what it is to do mathematics during lectures. Oehrtman’s (2009) analysis of students’ calculus metaphors indicated that their experiences likely shaped their metaphors. Works such as Oehrtman (2009), Pinto (2019), and Rupnow (2021) would suggest that as instructors teach similar content – quotient groups in this case – it should be expected that there is variation in the types of metaphors being used. Furthermore, students understanding of concepts can be conjectured to be tied to the metaphors provided during instruction. Although, we caution that the exact mechanism of this link at the undergraduate level is currently understudied.

Theoretical Perspective

In consideration of metaphors, it is often taught that a metaphor is a comparison between two things without using the word that like or as. However, modern linguistics considers this to be an antiquated and incomplete view of metaphor as it treats metaphor as a consequence of language. Rather, they take the view that metaphors involve more than language, but rather are tied to meaning and cognition. Despite disagreement among linguists in what constitutes a metaphor, they tend to agree that metaphor more broadly is a way of viewing one construct or object through the lens of another (Cameron, 1999). For the purposes of this study, in order to understand the concept of metaphor, we are drawing on the work of Lakoff and Núñez (1997, 2000) defining a metaphor to be a “cross-domain conceptual mapping” (Lakoff & Núñez, 1997, p. 32). Additionally, Lakoff & Núñez define two types of conceptual metaphors: grounding metaphors and linking metaphors. In this report, we focus on grounding metaphors. When using a grounding metaphor, a person is taking an experience from everyday life and mapping or projecting the experience onto the concept they are attempting to understand or communicate. We will be using this construct as a means to discuss the types of metaphors being used in teaching quotient groups and examining how quotient groups are understood from a metaphorical point of view.

Methods

The results of this study are a part of a larger dissertation study aimed at exploring quotient group instruction. Data collection took place across seven universities in the United States. The data examined here came from the first of a series of two semi-structured interviews in which the goal of the interview was to examine instructional practices surrounding the teaching and learning of quotient groups. A subset of the participants was purposefully chosen for this analysis based on their professed methods of teaching. Of the instructors selected, one utilized a research-based inquiry curriculum emphasizing partitioning, one primarily lectured and relied on physical manipulates and built up from ideas of cosets, while the last instructor primarily lectured and built quotient groups from more general ideas of equivalence classes.

Data Collection Methods

The goal of the initial interview was to examine instructional practices of those who teach abstract algebra with respect to their instruction on quotient groups. Prior to the discussion of teaching practice, each of the instructors were asked two questions about how they themselves conceptualized quotient groups and how they wanted their students to conceptualize quotient groups. The rest of the questions were focused on how they conducted instruction including questions regarding typical lessons, examples, proofs, assessments, and their view of their role in the learning process. Each of the interviews lasted approximately one hour and was videorecorded and transcribed. These responses were in turn used to inform the design of a follow up interview designed to illicit further discussion on decision making.

Data Analysis Methods

The first research open coded the transcripts (Braun & Clarke, 2006) identifying grounding metaphors. These metaphors were identified based on the participants' the use of "actionable words" or words that evoke a sense of action (i.e., the use of action verbs, gerunds, or participles) or physical classroom experiences that occurred. Once the initial list of grounding metaphors was generated, a second researcher read through each of the transcripts to offer critiques, pushback, and additional metaphors that may have been missed. Any disagreements were discussed between the researchers and worked through. The list of metaphors found in the results section reflects a subset of the agreed upon metaphors.

Results

We identified three overarching categories of metaphors. The first major class was quotient groups through the lens of construction-based metaphors. This metaphor class was further divided into building and demolition metaphors. Additionally, there were equivalence-based metaphors and positional metaphors. Table 1 provides a summary and definition of these metaphors. In the first column the metaphor class is listed along with examples of key phrases that fell into that class. A definition and example of each class is also provided in Table 1.

Table 1. Summary of Grounding Metaphors Class Used to Understand Quotient Groups

Metaphor Class	Class Definition	Class Example
Building Metaphors ● Building ● Clumping ● Sticking Together ● Partitioning*	A class of metaphors that describes the creation of quotient groups through building ideas	"we're doing these things and *building* these, these subsets"
Demolition Metaphors ● Partitioning* ● Collapsing ● Breaking/ Splitting ● Subdividing	A class of metaphors that describes the creation of quotient groups through demolition-based building ideas.	"…think about *collapsing* every one of those parallel lines on to this intersection point."
Equivalence Based Metaphors ● Setting equal to; setting to	A class of metaphors that grounds ideas of treating elements as if they are the same	"you're factoring out by Z mod 3, then you're looking at integers, and you can *ignore.* You can *ignore* multiples of

• Ignoring • Regarding as		three. Any things that differ by a multiple of three are essentially the same thing when you're looking at a quotient group.”
Positional Metaphors • Becoming • Going Around • Lying	A class of metaphors rooted in the idea of appearance or location. Often involve type of movement, but it is not necessary	“you think what would happen if you were to collapse every one of those circles to a point, what would you get each of those circles, is going to *become* one point.”

In the remainder of the results, we share some examples from three instructors to illustrate how these metaphors are used in their instruction descriptions and how the different types of metaphors support each other in grounding ideas of quotient structures.

Dr. A: Clumping and Collapsing

When explicitly asked how Dr. A wanted their students to think about quotient groups, they stated that they wanted them to understand quotient groups as “clumping.” In their explanation of what they meant by clumping, Dr. A stated

> So, this is a clock. So, this is going to be like Z_{12} you start with 0,1,2,3, 4, … ,11 and then you go 12, 1,2,3,4,5 so let's look at a quotient group of Z_{12} Mod four times Z_{12}. So, I start with Z_{12} and then, when I have four, I am back to zero. Go around again when I get to eight if I cut my string correctly to get back to zero. And so, then I’ve got Z_4, but I can see that all of these things, one I yeah and all of these are one item, and all of these are these are one. […] okay so I'm *grouping* or *clumping,* and I'm setting to zero.

During this illustration, Dr. A conveyed a building metaphor through the use of the word clumping. In this activity, they physically constructed piles of numbers (cosets) through this clumping or gathering action. To further explain this metaphor, they also drew on the positional metaphor of “going around” and equivalence-based metaphors in the use of the language such as “setting to zero” and “these are all one.”

As Dr. A continued to explain how they wanted their students to understand quotient groups and the ideas of clumping, they turned to another example and rather than using a building metaphor they instead used, “collapsing”, which is a demolition metaphor. Dr. A drew the image that is seen in Figure 1 and stated:

We’ll take the cosets of $y = x$ and it's always fun to watch them try to figure out what a coset is but they usually eventually get to the parallel lines which is correct. And so, all the cosets are parallel lines and now, when you think of this every one of those parallel lines is going to get clumped to one point, and so you can draw the counter diagonal and think about *collapsing* every one of those parallel lines on to this intersection point, and so you can see that if I take **R** cross **R** and mod by the diagonal, or, I guess, we do (x, y) such that x equals y. That you get something that is isomorphic to **R.**

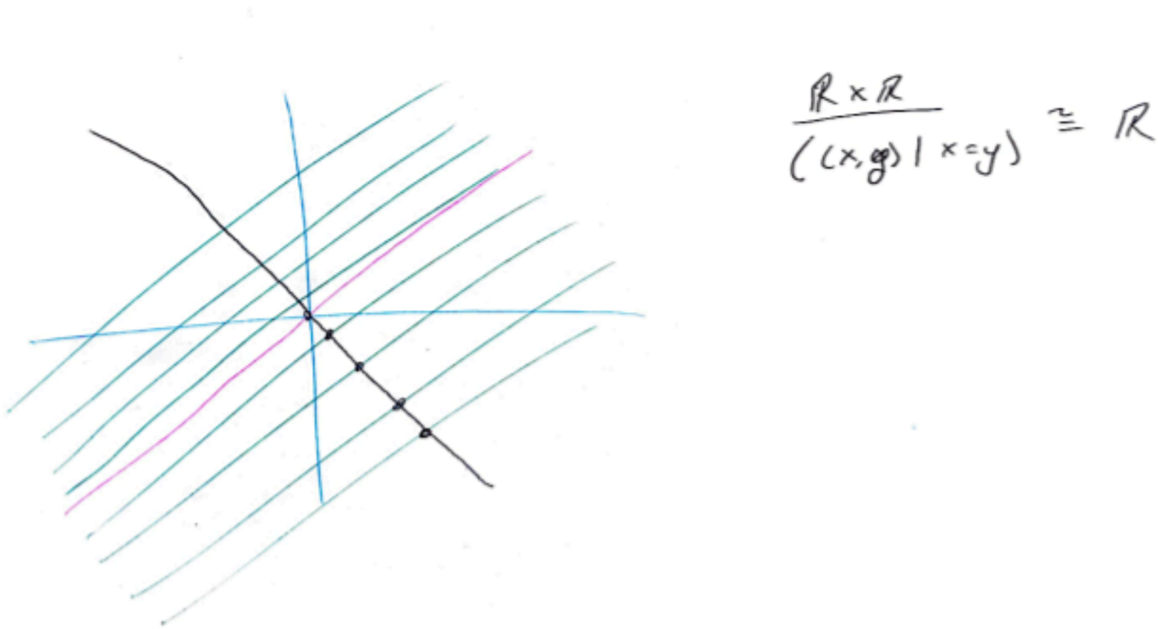

Figure 1

The main imagery that Dr. A utilized in this example is the idea that each of the parallel lines will fall or collapse. However, in this second except, we see that the collapsing is to a specific point thus furthering the metaphor of collapsing by giving a directionality of where the collapsing is going. It is worth noting that in this excerpt, the phrase clumped is used. However, unlike in the previous excerpt, “clumped” appears to describe the state of the parallel lines after they were collapsed rather than the action being applied to the parallel lines. Thus, this metaphor contrasts with the previous example as the mental imagery involved invoked different actions. In the first example, Dr. A invoked imagery of creating piles (and maintaining the subset structure with elements) whereas in this second example the collapsing moves from cosets as collections of elements to their role as single element losing the set structure through the act of setting to zero (emphasizing the representative element).

Dr. B: Partitioning to Build and to Deconstruct

Dr. B was similarly asked how they wanted their students conceptualize quotient groups and they explained that they wanted their students to understand quotient groups as “partitioning” stating "... so this idea of partitioning I think is powerful for thinking about the structure where you're building this, these weird things that are sets and elements kind of at the same time." In this explanation of partitioning, Dr. B communicated that partitioning was a way of building. This class of building continued in their explanation as they made statements such as "building up cosets" and "understanding the machinery involved in building."

In a similar manner to Dr. A, Dr. B also made a transition from building metaphors to demolition construction metaphors. However, unlike Dr. A, the demolition metaphor connected to the same action metaphor: partitioning. While explaining an activity regarding the partitioning of D_4, Dr. B stated

> Is there a way that we could *partition* up this D_4 table into something that acts like evens and odds, right? […] I always expect that the students are going to just *break it up* into like the rotations and the flips. […] I always have groups of students who break it up in other ways, but. They always break up in ways that that work. They're doing something really reasonable, and how they're subdividing these up, and it's now kind of laying this foundation that like one of these subsets has to act kind of like an identity type of thing, like the evens did. We're seeing every element end up somewhere […] This is kind of emphasizing the partitioning into new elements. And that physical color gives you something that you can think of this all as like one object over altogether. So, we spent a lot of time doing that.

In this explanation, Dr. B is described the creation of quotient groups through actions such as "subdividing" or breaking up. However, this is still describing the understanding of partitioning because they first posed the task of partitioning the group D_4, yet they hold the expectation that students will break things up. Thus, this explanation yielded partitioning as a demolition metaphor in this case, but ultimately as both a demolition and building metaphor as it had previously been described as building. Additionally, within this explanation, Dr. B drew on the use of a positional metaphor through the use of the phrase "end up somewhere" as they are discussing this transformation of the elements within the original group structure. In this positional metaphor elements are moving from the original group to their assigned coset.

Dr. K: Equivalence-Based Metaphors

Dr. K emphasized quotient groups as an extension of equivalence classes. In looking at Dr. K's conception of quotient groups, he stated:

> It's when you're factoring out by something. It's that's a factor you can ignore. So simplest examples of course, are modular arithmetic. So, if you're factoring out by Z mod 3, then you're looking at integers, and you can ignore -- you can ignore multiples of three. Any things that differ by a multiple of three are essentially the same thing when you're looking at a quotient group.

Within this explanation, Dr. K used primary metaphorical expression: ignore. This expression is used to convey the mathematical idea of equivalence and that "you have a set of objects" in which things are regarded as the same. Dr. K further elaborated on these ideas as they explained one of the examples they liked to use in their practice.

> I do use examples which are not which are not finite examples. Things like the fact that the nonzero complex numbers are a group under multiplication. S^1 is a subgroup, so then what is S^1 this time? Meaning the unit circle, of course. And what is the nonzero complex numbers mod S^1, right? And what is? And also, the positive reals are a subgroup, so what's the nonzero complex numbers mod the positive? These are very geometric things

> for them to think about. And modding out by S^1 really mean it also reinforces this idea. Well, now we're regarding two nonzero complex numbers as the same if they lie on the same circle about the origin and your quotient group, then is effectively the positive risk, so I like to talk about those examples somewhat.

Additionally, in this example, sameness was used in conjunction with the positional metaphor "lie." Unlike the previous positional metaphors, lie takes on a more static nature in that it lacks any type of movement. However, it is an important metaphor to note in that it serves as a justification for how Dr. K judged sameness in this example in that two points were considered to be the same if they were on (*lie)* on the same circle.

Discussion

We shared data from these three instructors because during their description of their teaching of quotient groups, they each seemed to emphasize a different aspect of quotient group understanding. During our analysis, there were four primary classes that emerged as instructors discussed quotient groups: building, demolition, equivalence, and position. As we considered these metaphors and the use of them, we hypothesize that each type of metaphor may reflect different ways of thinking about quotient groups. Dr. A emphasized *clumping* and *collapsing.* These types of actions emphasize an outcome where elements are clumped into sets then collapsed into a single element. These actions may reflect the duality between elements and sets. Dr. B's metaphors were rooted in *partitioning* from original group to quotient group traversing the duality between elements being members of the original (*subdividing)* and these elements now being cosets in the quotient group *(building).* Finally, Dr. K leveraged ideas of equivalence to emphasize *ignoring* which may reflect the way that a coset can be thought of in terms of its representative element. In some ways these differences may emphasize the different ways of conceptualizing Zn elements as subsets, set with representative element, and representative element only.

For the scope of this paper, we focused on the primary metaphors shared by these instructors. In all three cases the primary metaphors focused on the element and set relationships rather than the group operation. We also note that positional metaphors could be found in all three instructors' language but did not appear as prevalent. Other metaphor classes that appeared in the data but that are not discussed in this report include the personification of normality and its role in "breaking" or "busting" groups or maintaining "machinery." Future research will involve testing this taxonomy with other instructional interviews and expanding classifications to include other elements of quotient groups such as normality. Additionally, future work will also examine the use of a second type of metaphor - linking metaphors - that instructors use as they discuss quotient groups as these were at times used in tandem with some of the grounding metaphors that the instructors were using. Both aspects of this future work will be carried out with the remaining thirteen participants within the larger dissertation study.

At this point, we conjecture that during quotient group instruction there is a lot of imagery that is at play due to the language that instructors use. However, students may or may not be influenced by their instructor's metaphor. Finally, we note that it is unlikely that quotient groups can be conveyed fully with any particular metaphor or class of metaphors. The language used by these instructors reflects attention to a number of salient dualities and therefore grounding metaphors for quotient groups are likely needed in clusters to fully account for the concept.

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