To Do Calculus, We Need to Specify R: Lecture Meta-stories & their Underlying Assumptions.

Anna Zarkh
University of California, Berkeley

This papers examines the stories two instructors tell about why formally defining real numbers in the context of a Real Analysis course is a desirable objective. While the two stories exhibit some variations, the analysis shows that both narratives relied on the following constitutive elements: (1) calculus is a desirable activity, (2) students' current understanding of R is lacking, and (3) to use R for calculus, there is a need to specify it precisely. I discuss how these assumptions can be misrepresenting of practice and experienced as problematic impositions on students.

Keywords: real analysis, real numbers, meta-mathematics, narratives, values.

The stories we tell about the nature and purpose of mathematical practice matter. They frame mathematical activity in and out of the classroom, render it sensible (or not), and help delineate what actions are deemed appropriate, valuable or even feasible in a given context (Schoenfeld, 1989). Stories about math can be problematic. Many mainstream stories, such as "there is only one math" (Hersh, 1991) or "math is a young man's game" (Barany, 2021), both misrepresent disciplinary practice and alienate students by constructing a world that few can see themselves in. One context for meta-mathematical story-telling is introductory proof-based university courses, such as Real Analysis (RA). In such courses, instructors may feel the need to explicitly explain and justify the course and its new way of doing math.

RA is a challenging course in the undergraduate curriculum, largely due to the novelty of its proof-oriented epistemic game. Dawkins and Weber (2017) suggested that the norms and practices of this epistemic game are difficult to learn in part because the underlying values and purposes these norms uphold are not made explicit in instruction. However, we have little empirical evidence about whether and to what extent instructors address such meta-issues in lectures. This paper addresses this issue, by examining how instructors describe and justify real analysis in the first lectures of their RA courses. In particular, I address the following research question: What meta-stories about the nature and purpose of RA do instructors tell in lectures?

This study is informed by sociocultural theories (Wertsch, 2012). Mathematics is conceptualized as a discourse that involves telling stories (Sfard, 2008). Here, I focus on meta-stories, which I define as stories *about* mathematical practice. I refer to these as stories, rather than instructors' beliefs, to highlight their contingent and situated nature.

Methods

This study is part of a larger video-based micro-ethnography (Derry et al., 2010; Erickson, 1992) of undergraduate RA lectures taught in Fall 2020 at a large public research university in the US. All lectures were delivered online, over Zoom, due to covid-19. Primary data for the larger study include video recordings of all lectures, and auxiliary documents such as lecture notes. Four instructors (Alex, Cai, David and Emmett) participated in the study. All four worked as research mathematicians at the time of data collection. Alex, Cai and David were early career scholars and Emmett was a senior faculty member in the department. For this study on meta-stories, I consider only the first lecture of each instructor (Alex, David & Emmett, 50 min; Cai, 90 min).

Data Analysis

Data analysis proceeded in two steps. First, I generated secondary data through transcription, segmentation and descriptive coding of episodes. Then, I coded episodes for meta-talk, consolidated into story themes, and iteratively constructed summary descriptions for each theme.

Transcription, segmentation & descriptive coding. Each of the introductory lecture videos was transcribed for talk and salient gestures. Transcripts were segmented into short episodes, of 1 min length or less. Segmentation was based on "natural" transitions. These included linguistic markers (e.g. "So" and "Okay"), and shifts in topic or inscriptional focus. Each 1 min episode was given a title that served as its descriptive summary, using a mixture of content and in-vivo coding (Saldaña, 2021). I then grouped the short 1 min episodes into larger coherent wholes, according to the overall type of classroom activity (e.g. if episodes 3-8 were all part of a single proof, they were grouped together under the heading "proof of …"). This multi-step process resulted in a hierarchically structured outline of each lecture's transcript (Erickson, 1992). The structured transcripts were the data sources used for subsequent analysis.

Coding meta-talk & meta-story summaries. Meta talk appeared in lectures in two ways. There were entire episodes devoted to meta talk, and there were shorter instances of meta-talk (e.g. a single sentence) that occurred in the midst of an episode with a different focus. The coding process was inductive and proceeded as follows. First, I flagged episodes of meta-talk that addressed either what the course or the sub-field of RA are, or why they are worth studying or doing. To determine whether an episode or utterance counted as meta-talk, I relied on explicit linguistic indicators such as "our goal in this course" and "Real Analysis is." I then went through each episode that was flagged for meta talk, and created sub-codes to capture different story types. Shifts to different stories were often marked by the instructors themselves (e.g. "another goal of this course is…"). Through an iterative process of refinement, I delineated distinct categories of meta-stories about RA that together account for all meta-talk episodes and instances. With these meta-story categories, I went through the structured transcripts again, looking for confirming and disconfirming evidence and more instances of meta-talk. This process led to further refinement of the codes and consolidation in the form of story summaries.

Findings

All four instructors devoted class time to meta-talk about the nature and purpose of RA during their first lecture. Across the data set, I identified the following five meta-stories about RA:

1. To do calculus, we need to specify R (Real numbers). Our current understanding of R is vague. In RA, we define R precisely and build calculus from it.
2. RA is calculus with proof. Since calculus is known, RA is good for learning to prove.
3. Calculus is a tool that sometimes breaks. RA is a theory of how the tool work. It is good to learn the theory for future tool use and new tool development.
4. Mathematicians made mistakes in calculus and found solution in rigor. We follow them.
5. RA is a theory of connections between calculus and more fundamental topics. It is good because it makes calculus simpler and more elegant.

The table below (figure 1) shows the presence or absence of the five narratives in each of the instructor's lectures. In this paper, due to space limitations, I elaborate only on the first story: "to do calculus, we need to specify R." In the remainder of the findings section, I will describe how Emmett and David each constructed a version of this story in their first lecture, and then compare their version across key underlying assumptions: (1) calculus is a desirable activity, (2) students' current understanding of R is lacking, and (3) to do calculus, there is a need to specify R.

	Alex	Cai	David	Emmett
S1: precise R			precise R	precise R
S2: learn to prove	learn to prove	learn to prove	learn to prove	
S3: fix calculus	fix calculus	fix calculus		
S4: follow community		follw community		
S5: connections	connections			

Figure 1: Appearance of meta-stories in each instructor's first lecture.

Meta-story 1: To Do Calculus, We Need Precise R.

Emmett's meta-story. Emmett discussed the two constitutive terms of the course title, 'real' and 'analysis,' separately. Starting with 'analysis,' he offered the following definition:

> … an informal and incomplete definition of analysis is that analysis is that part of mathematics in which limits are used to solve problems. So, the key concept in analysis is that of a limit. I would say maybe the second concept is that of an inequality. But maybe the most fundamental concept is the limit. And that's really what this course is about. Limits and closely related concepts. (episode 10)

In the above quote, Emmett defined analysis as a subfield of math, characterized by its focal concepts (limits, inequality) that are framed as tools for solving problems.

Next, he shifted attention to the other part of the title, namely, the adjective 'real.' He posed the question "so, what about real numbers?" and shared a slide with the following famous quote from the German mathematician Leopold Kronecker: "God made the whole numbers; humans made the rest." Emmett interpreted the distinction Kronecker's quote makes between god-made and man-made numbers as implying that the real numbers are "not as fundamental as the whole numbers," and declared that the course will take that "point of view."

On the next slide, he elaborated on what makes real numbers less fundamental. He asked "what is a real number?" rhetorically and provided three "informal descriptions" as answers: real numbers are (1) points on the number line ("when you teach a calculus class, you draw a line on the board and you say here are the real numbers"), (2) expressible using decimals ("Any number you can write like that, positive or negative, is a real number"), or (3) possible values of a variable x in a calculus course. Emmett then explicitly framed these descriptions as inadequate:

> So that's obviously not a very rigorous notion, but that captures what we're talking about. I hope that you'll all agree with me that none of these answers are anywhere near as clear and unambiguous as the positive whole numbers. We all are confident that we can start with one, add one to it, add one to it, add one to it, and keep going. And we more or less understand what sort of objects we're getting that way. The real numbers are a bit more mysterious. (episode 12)

In the perspective offered above, familiar descriptions of real numbers (standard in calculus courses) are declared problematic because they are "informal" and not "very rigorous." The ambiguity and mystery of real numbers is justified through a comparison to the presumed clarity with which whole numbers are comprehended. According to Emmett, whole numbers as mathematical objects are clear and unambiguous; by imagining repeated addition, "we can more or less understand what sort of objects we're getting." Emmett uses this appeal to a presumably shared experience of different degrees of ontological clarity to justify assigning real and whole numbers a different epistemic status: "And in this course, we will not take basic properties of the real numbers for granted. Unlike these more basic kinds of numbers."

In Emmett's story, filling this ontological and epistemological void is a key problem the course aims to solve. Accordingly, constructing the real numbers in a clear an unambiguous way played a central role in "the program" Emmett laid out next:

> Instead, we're going to spend some time and we're going to use the rationals, which we've agreed, we do understand that we can assume things about. We're going to use the rationals to construct some objects. And this collection of objects will have all the properties that the real numbers ought to have. And then we'll just call these objects the real numbers. And as the course goes on, we'll use them. We'll use them, we will develop the concept of a limit. And we'll use them with, together with limits to develop the theorems that underpin calculus and to solve various other kinds subproblems. Ok, so that's the program. (episode 13)

The "program" Emmett articulated describes RA as sequential coverage of mathematical content (R, limits, calculus), where each step is 'built' on top of the previous one: rational numbers are used to construct real numbers, real numbers are used to develop limits, limit are used to develop calculus theorems. Analysis is situated later than real numbers in the sequential development of content. But, how are these two connected? Why are real numbers needed for doing analysis, assuming the latter is an agreed upon goal? Later in the lecture, Emmett made additional meta-comments, that can be seen as filling this gap in the story:

> … the rationals are not complete. And that, in a nutshell, is why this course is real analysis instead of rational analysis, the rationals are defective from our point of view. We want to be able to take limits, completeness, the rationals are not complete and there will be instances when we can't take limits when we want to. So the rationals don't work for us and we need the reals. (episode 34)

Here, the course's focus on real numbers is justified by their necessity for doing limits. In this argument, we don't care about real numbers *just* because they are unclear and complex. We care about them because we want "to take limits," and rational numbers do not always allow it. The problem is that rational numbers, unlike real numbers, "are not complete" which makes them "defective from our point of view."

In a later meta-comments, Emmett repeated the course's program description, adding detail:

> … we're going to construct a set of objects that has all the properties that the real numbers ought to have. We'll just call that set the real numbers, and then we'll use it together with limits to solve problems and to justify the foundations of calculus. OK. And the objects that we construct are called cuts, or they are called Dedekind cuts after, after a mathematician named Dedekind. (episode 43)

In this last comment, Emmett provided additional information. Namely, that the to-be constructed real numbers are "objects" called Dedekind cuts. He immediately followed this programmatic outline with a comment about the resulting ontological status of real numbers:

> So notice that this approach sidesteps the basic question of what the real numbers actually are. And, you know, that's some sort of metaphysical question that doesn't really have a mathematical meaning. So, we're simply not going to go there. We're simply going to construct something and we'll call it the real numbers. Because of the theorem I told you at the end of the last set of slides, that's a reasonable approach. Any fully mathematical question you can ask will have the same answer for our real numbers, as for anybody else's real numbers. (episode 43)

In this last quote, Emmett clarified that the course's "program" will not provide an answer to the ontological question of "what the real numbers actually are." Despite the fact that he used the (presumed) ontological ambiguity of real numbers to motivate the program at the beginning of the lecture, in this later comment, he framed the ontological question as "metaphysical" and devoid of "mathematical meaning." To justify this dismissal of the original question, he referenced the theorem that any two complete, Archimedean, ordered fields are isomorphic.

Based on this meta-talk, I summarized Emmett's narrative as follows: *We are interested in doing analysis, which is using limits to solve problems. To do limits, we need real numbers. But, we don't know what real numbers are in a clear and unambiguous way. We do, however, clearly understand natural, whole and rational numbers. So, the course starts with constructing [something we call] real numbers from rational numbers, and develops calculus from that.*

David's meta-story. David began his meta-story by rhetorically posing the question: "what's the goal of this course?", and immediately providing the following answer:

> This is introduction to analysis or real analysis. So what we're <u>really</u> trying to do is come up with a set of tools to rigorously study what happens on… what the real numbers are and how the real numbers behave. (episode 9)

The above description features some of the themes we saw in Emmett's presentation. David claimed that the course's objective ("what we're really trying to do") is to develop "tools." The tool metaphor is similar to Emmett's framing of limits as something that can be used to solve problems. However, unlike in Emmett's story where the nature of the problems remains underdetermined, in David's description the objective is characterized as the exploration of a mathematical reality in which real numbers are the landscape and focal characters: "what happens on [the number line]" "what real numbers are," "how they behave." At this point in David's story, understanding phenomena pertaining to real numbers is the goal, whereas in Emmett's story, it was a necessary step in a 'program' whose ultimate objective was calculus, i.e. 'using limits to solve problems'. But, this was just the start of David's story. He continued:

> … the core ideas in this course, and the core themes we are going to explore are. Well, I guess one core theme is essentially approximation. And the real question we have, right? You know you have a vague notion, probably sort of what a real number is, but if you <u>actually</u> want to specify what a real number is, it can be a little tricky. (episode 9)

David mentioned approximation as a theme but did not connect it to the declared focus on real numbers. It seemed as though he was about to list several core ideas, not just approximation, but he quickly shifted to what he called "the real question," which is that of specifying real numbers. Similarly to Emmett, David problematized real numbers by invoking a binary opposition between two epistemic stances: having a "vague notion" versus "actually" specifying. This binary allows seeing the specification of "what a real number is" as a problem to be solved.

David proceeded to justify why specifying real numbers is useful. Recall that in Emmett's story the only justification is the purported experience of ambiguity (for David, "vagueness") with familiar descriptions of real numbers. David's narrative features more rationales. Specifically, he claimed that specifying real numbers is necessary for more advanced calculus:

> … if we want to start doing complicated things like calculus or analysis of things in higher dimensions or working on- or doing calculus on a surface that isn't actually the real line. If you want to do calculus on the surface of a sphere, for example, it gets a little bit tricky and it becomes important to know precisely what sort of thing you're talking about. And so the goal of this class is to sort of lay that groundwork that dealing with the real numbers are, so that we can then in the future expand it to talk about more interesting, more interesting topics. (episode 10)

Similarly to Emmett, David invoked a building-foundation metaphor ("lay the groundwork") to interpret the significance of 'specifying real numbers' as a base step in the development of a hierarchically organized system of mathematical knowledge. However, unlike in Emmett's story where the end point was 'regular calculus,' here the motivating horizon is more advanced versions of calculus – calculus in higher dimensions or on surfaces such as a sphere – that are "interesting," yet "tricky" and "complicated." Thus far in David's story it is the complexity of these more advanced topics, rather than any difficulty with regular calculus, that necessitates an ontological specification of R ("it *becomes* important to know precisely what sort of thing you're talking about"). He developed this idea further in relation to infinite dimensional calculus:

> You might be curious. Can you do something that looks like calculus in infinitely many dimension? I mean, you've probably taken- I'm sure one of the courses you've taken as a prerequisite to this is a multivariable calculus class that probably did two or three, maybe n-dimensional calculus. But can you have meaningful kinds of calculus in infinitely many dimensions? And the answer is yes. But developing that, there are many, many subtleties to get through to develop that. So we're going to start somewhere to have something to build on. And so we really want to make these ideas of <u>what</u> makes the real numbers, what they are, precise. (episode 10)

We see here a repetition of the same theme. Doing more advanced calculus (in this case, infinite dimensional) is subtle and complex. Mathematical knowledge is a hierarchical structure. To be able to (in the future) cope with this complexity of more advanced levels, it is necessary to do preparatory work at the base ("start somewhere to have something to build on"). In particular, the prep work involves specifying real numbers. Unlike Emmett, David did not provide an explanation for why real numbers, specifically, is the "somewhere" where one "starts."

I summarized David's narrative as follows: *We are interested in doing advanced calculus, such as calculus on surfaces, and calculus in multiple and infinite dimensions . To do advanced calculus, we need precise real numbers. Our current understanding of real numbers is vague. So, the course focuses on specifying real numbers, to have something to build on.*

Summary. I compare Emmett and David's meta-stories across constitutive assumptions.

The first assumption is (1) students are interested in doing calculus. Emmett treated interest in calculus as a taken-for-granted goal, both in defining analysis and in positioning calculus theorems and limits as the end goals of content development. In David's meta-story, the objective is not regular calculus, but rather more advanced versions, such as calculus on surfaces and in infinite dimensions. But, like for Emmett, the interest in such topics is assumed and positioned as an ultimate goal. A RA instructor may reasonably assume that a motivation for calculus – What kinds of problems do limits help solve? Why should we care? – was addressed in previous courses. However, David's framing of interest in infinite dimensional calculus as something one is naturally curious about after doing finite dimensional calculus can be problematic. It is historically inaccurate and potentially alienating, as few students are likely to see themselves reflected in this positionality of wondering about artificial extensions of theory.

The second assumption is that (2) students' current understandings of real numbers are lacking. Emmett conveyed this by listing descriptions of real numbers students are familiar with and framing them as "not very rigorous," unclear, and ambiguous. David, similarly, claimed that students only "have a vague notion … of what a real number is." In both stories, the negative epistemic stance is not fully justified. Emmett's use of phrases such as "obviously" and "I hope that you'll all agree," actually highlights the absence of an explanation. Indeed, a negative epistemic and ontological stance toward real numbers is not self-evident. Students (and various professionals, including mathematicians) have been successfully using 'informal descriptions' of real numbers for a long time (e.g. in calculus) without feeling the need to specify them in the way Emmett and David suggested. To date, in many (if not most) contexts of doing calculus, people unproblematically rely on representations of real numbers which Emmett and David labeled as ambiguous and vague. Thus, the offered stance can be experienced as an imposition.

The final assumption is that (3) to do calculus, R needs to be specified. Emmett did not provide explicit justification for this claim, but the programmatic description he offered relies heavily on building-foundation metaphors for mathematical knowledge, and within such a metaphorical conceptualization of math it may seem as self-evidently true that concepts need to be 'built on solid ground' to be viable. David's story incorporated building-foundation metaphors too (e.g. "lay the ground work"). However, David offered an additional rationale by positioning the specification of R, not as something done for its own sake, or as something needed for doing regular calculus (that may directly contradict students' experiences of doing calculus without specifying R), but rather as something necessary for doing more advanced calculus, one that students have not yet seen. This story defers a compelling motivation for specifying R to a future context (e.g. infinite dimensional calculus). The fact that students are tasked with 'taking David's word for it' further highlights the absence of an accessible rationale.

Discussion

Mainstream stories about math are rife with idealizations of the discipline that both alienate many learners and do not tell the full story of what the practice is like and what, and for whom, it is good for (Hersh, 1991; Wagner, 2022). This paper aims to contribute to our understanding of how such idealizations are constructed in the gatekeeping educational context of RA lectures. Critically examining the assumptions instructors' meta-stories rely on and how these assumptions relate to past and current professional practice and students' past curricular experiences, can help us craft meta-stories that are both realistic and more compelling. The analysis also illustrates that instructors *do* address axiological issues in lectures (Dawkins & Weber, 2017), though the effect of their stories on students requires further examination.

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