# "Close but not Exact": An Alternative to Introducing Derivatives

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*In this paper, I report on the findings of two students engaged in a teaching experiment on instantaneous rate of change (derivatives). While derivatives are a quantification of how two quantities covary, many students tend to have static images of the derivative (Zandieh, 2000). This teaching experiment was designed to support the students in building a meaning for instantaneous rate of change based in quantitative and covariational reasoning. The results of this study provide empirical evidence of the benefit of alternative teaching methods to the current standard curriculum.*

*Keywords:* Derivative, Quantitative Reasoning, Teaching Experiment

Research on Calculus education has been extensive and covers many topics. In particular, there is a large body of research on the learning and teaching of derivatives and instantaneous rate of change (e.g., Orton, 1983; Ferrini-Mundy & Graham, 1991; Zandieh, 2000; Yu, 2020). One typical method of introducing the derivative concept involves sliding secant lines towards one of the endpoints to get continuously better approximations of a tangent line, the slope of which represents a quantity we call *instantaneous rate of change*. However, researchers have indicated that students often struggle with connecting this sliding secant line imagery with rate of change (Ferrini-Mundy & Graham, 1991; Zandieh, 2000; Ubuz, 2007). This idea of quantities changing continuously and covarationally is central to the idea of derivative, yet the research is clear that students often do not associate quantities as covarying when they think about derivatives (Zandieh, 2000; Byerley et al., 2012; Yu, 2020). Due to these issues, this manuscript reports on the results of a teaching experiment designed to support students in developing a productive understanding of instantaneous rate of change.

## Theoretical Background

If we examine the limit definition of derivative, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$, the derivative involves an average rate of change between the input values $x$ and $x + \Delta x$ ($\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$) where the variation in the value of $x$ gets smaller and smaller ($\lim_{\Delta x \to 0}$). In the standard presentation, where a sliding secant line converges to a tangent line, students need not attend to what the average rate of change describes about a given context. This is evident in the literature (Zandieh, 2000; Byerley et al., 2012) when students interpret the value of a derivative (e.g., $f'(3) = 7$), as being only associated with the slope of a tangent line and not necessarily attending to how quantities may be covarying. Therefore, it stands to reason that students should simultaneously attend to $x$ and $f(x)$ covarying and how choosing smaller and smaller input intervals can net us what we call an instantaneous rate of change.

## Relevant Literature

Ely and Ellis (2018) proposed that leveraging what they call *scaling-continuous covariational reasoning* could support Calculus students in developing productive ways of thinking. Building off this, Ely and Samuels (2019) provide empirical evidence of alternative ways of teaching the idea of derivative with their "Zoom in Infinitely" method. These studies indicate that supporting students in reasoning about how two quantities' values covary together can help them better understand the derivative concept. This study provides another example of teaching one aspect of the derivative (the limiting portion) using an alternative method that deliberately focuses on supporting students in reasoning quantitatively. Compared to the works of Ely and Samuels, the students are not zooming in on an existing graph of a function. Instead, the students create a piecewise linear function and continually choose smaller and smaller intervals to model an object's movement. This study's research question is: What understanding of the derivative concept do individual students develop during an instructional sequence designed to support students' quantitative and covariational reasoning?

## Methodology

This study was part of a larger study conducted by engaging students in individual teaching experiments (Steffe & Thompson, 2000). The students selected were enrolled in a Calculus 1 for Engineers course before learning about secant lines and instantaneous rate of change. The teaching experiment involved six sessions (1 pre-interview and 5 teaching sessions) that focused on characterizing and advancing students' ways of thinking about rate of change. This document reports on two students in the fourth teaching session, where the teacher worked with the students to develop a basis for a productive understanding of instantaneous rate of change. In this session, the way of thinking that the instructor wanted students to develop was the idea that "One can obtain better approximations of the (instantaneous) rate of change at a given value of the input variable by determining average rates ($\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$) of change on smaller and smaller intervals ($\lim\limits_{\Delta x \to 0}$) that include that value of the input variable."

## Background

Before the fourth teaching session, the third session involved having the students program (writing a piecewise linear position function) to model the movement of a runner by choosing smaller and smaller intervals (Yu, 2023). That session was primarily focused on developing a productive meaning for average rate of change as an imagined constant rate of change that would provide the same net changes in the output quantity over the same input interval. Additionally, it also seeded the fundamental idea of this paper by providing an animation of their programmed function that showed their model's movements as getting more accurate to the actual runner (Figure 1).

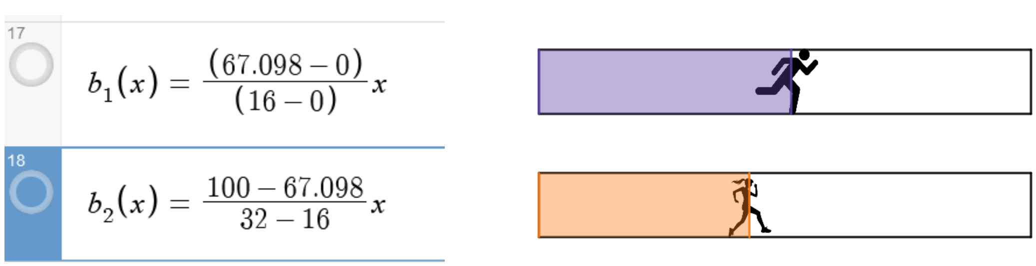


*Figure 1: Example of Programming the Bottom Runner using the Average Speed of the Top Runner*

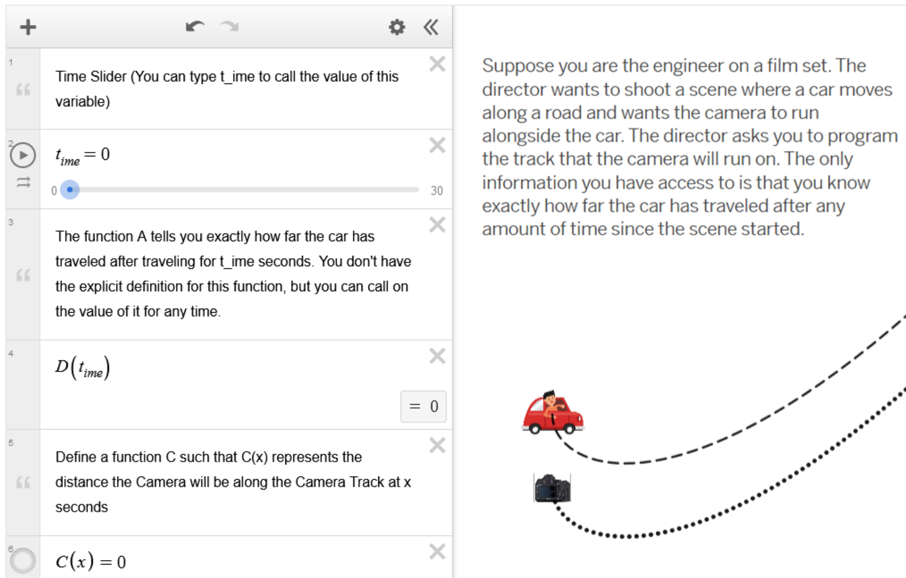


*Figure 2: The Camera Task*

In the fourth session, students worked through a Desmos Applet and were tasked with modeling the motion of a camera to match a car's motion (Figure 2). While this task is similar to the third teaching session, one important note is that the Desmos Applet provided no indication to construct a piecewise linear function using average rates of change. Instead, students were prompted to discuss their thinking and how they would attempt to mirror the car's movement. Later in the session were questions on how they might determine the speed at a particular time value and approximate future values of a function using that speed.

**Teaching Session 4 – Scott**

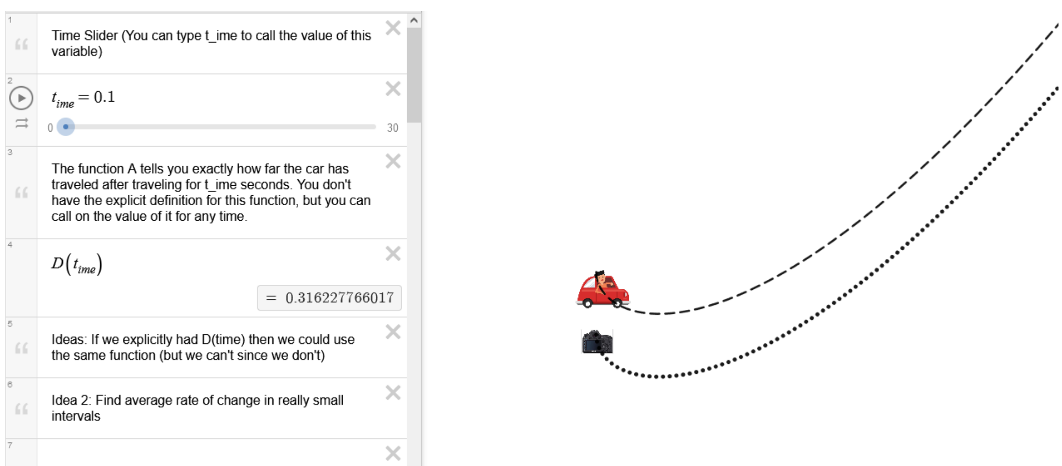


*Figure 3: The Camera Problem and Scott's initial ideas*

The camera problem prompted students to program a camera's movement on a track to mirror the movement of a car on a parallel track. Scott initially stated that he wanted to determine the equation of the line that would create the same path the camera was on. In other words, Scott wanted a function where the physical shape of that function's graph matched the shape of the track he saw on the Desmos applet. However, he realized that having the function whose graph matched the physical shape of the track path would not model how the camera's distance traveled would vary as time varied. Scott then suggested using the given distance function of the car, $D$, but soon recognized he could not do this since he did not have the function definition for $D$ (Figure 3). Scott then decided that we could "find the average rate of change in really small intervals where it would be pretty close but not exact." As Scott started to define a function, $C$, where he used the car's average speed and multiplied it by an amount of time, he verbalized that he wanted to use the same method as the Runner Task (Figure 1) by using average rates of change. Scott's choice to utilize average rates of change is essential to note since the questions accompanying the Desmos applet did not have any prompts to define piecewise linear functions or use average rates of change to model the camera's motion. Scott broke up the 30-second interval into one-second intervals (Figure 4). As he defined his piecewise linear function, he checked his work by playing the animation to ensure that the camera moved according to his expectations.

| $t_{IntNumber}$ | $m$ | $d_{car}$ |
|---|---|---|
| 1 | $\frac{D(1)-D(0)}{1}$ | 0 |
| 2 | $\frac{D(2)-D(1)}{1}$ | $D(1)$ |
| 3 | $\frac{D(3)-D(2)}{1}$ | $D(2)$ |
| 4 | $\frac{D(4)-D(3)}{1}$ | $D(3)$ |
| 5 | $\frac{D(5)-D(4)}{1}$ | $D(4)$ |
| 6 | $\frac{D(6)-D(5)}{1}$ | $D(5)$ |
| 7 | $\frac{D(7)-D(6)}{1}$ | $D(6)$ |
| | 19 more rows Show all | |
| 27 | $\frac{D(27)-D(26)}{1}$ | $D(26)$ |
| 28 | $\frac{D(28)-D(27)}{1}$ | $D(27)$ |
| 29 | $\frac{D(29)-D(28)}{1}$ | $D(28)$ |
| 30 | $\frac{D(30)-D(29)}{1}$ | $D(29)$ |

*Figure 4: Scott programming the camera's distance traveled over time using the car's average speed in each one-second interval*

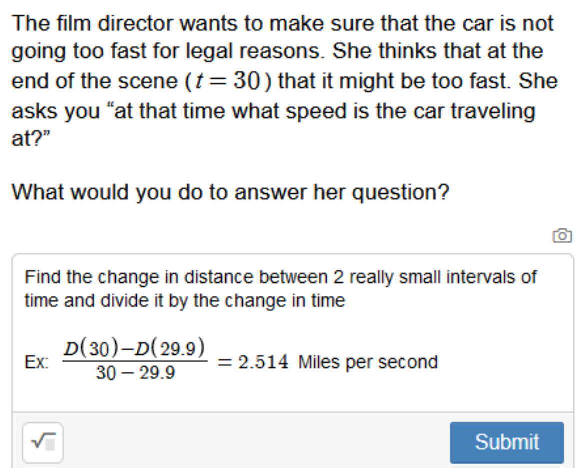


*Figure 5: Scott's response to finding the speed of the car at a given input value*

After programming the function to model the camera's distance traveled along the track with respect to time elapsed, Scott was asked a series of questions on determining the car's speed at a particular value of time (Figure 5). Scott stated that he would "find the change in distance between 2 really small intervals of time and divide it by the change in time." He later generalized this statement: "I would just take 2 values very close to that time, and I would calculate the rate of change between them." There are several aspects of Scott's worth response worth highlighting:

1. The sizes of the time intervals he chose to determine the speed at $t = 30$ (Figure 5) were different from the size he used to program the camera's motion (Figure 4). For

programming the camera's distance traveled, he used the car's average speed in 1-second intervals, and for the speed at $t = 30$, he used the car's average speed in a 0.1-second interval. This suggests that Scott's choice in the "2 values very close to that time" was arbitrary and that no specific interval size was needed.

2. Scott verbalized that using average rates of change over a small time interval "would be pretty close but not exact" to modeling the car's motion as time passed. As he answered each of the prompts about determining the car's speed at a particular time, he never verbalized a need to justify that they were close approximations. It is likely that because the animation displayed the motion of the camera and the car was essentially the same; he might have believed that his approximations were good enough and that the difference between the actual and the approximated distance functions was insignificant.

**Teaching Session 4 – Hans**

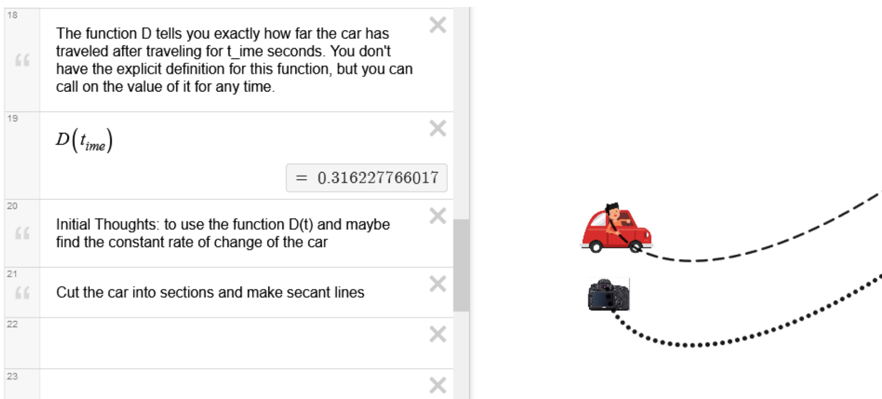


*Figure 6: The Camera Problem and Hans' Initial Ideas*

Like Scott, Hans' initial approach to the camera problem was to determine the equation of the line whose graph matched the path of the road. He then suggested finding "the constant rate of change of the car." However, after playing the animation, Hans recognized that the car was moving at varying speeds, so he proposed that we "cut the car into sections and make secant lines" (Figure 6). Hans verbalized that this was similar to the third teaching session, and he wanted to build a piecewise linear function using the car's average speed over small time intervals to model the camera's motion.

Hans employed 3-second intervals (Figure 7) and did not play the animation until he finished defining the piecewise function. After playing the animation, he noticed that the camera and the car's movements did not align adequately in the first few seconds. He then suggested, "we could do better if I made it like 1 second [the interval size] instead." Rather than having Hans rewrite his piecewise function, the interviewer asked hypothetical questions such as "suppose we wanted to do 1-second intervals, what would we need to change?". Hans replied that he would have to change everything since "you wouldn't be able to use like 3 seconds…like how these are done. I would have to re-do them to resize these rates of change." After this round of questioning, Hans demonstrated that he could obtain better approximations of the car's distance traveled with respect to time elapsed by using average rates of change over a smaller time interval.

| $t_{IntNumber}$ | $m$ | $d_{car}$ |
|---|---|---|
| 1 | $\frac{(D(3)-D(0))}{3-0}$ | 0 |
| 2 | $\frac{D(6)-D(3)}{6-3}$ | $D(3)$ |
| 3 | $\frac{D(9)-D(6)}{9-6}$ | $D(6)$ |
| 4 | $\frac{D(12)-D(9)}{12-9}$ | $D(9)$ |
| 5 | $\frac{D(15)-D(12)}{15-12}$ | $D(12)$ |
| 6 | $\frac{D(18)-D(15)}{18-15}$ | $D(15)$ |
| 7 | $\frac{D(21)-D(18)}{21-18}$ | $D(18)$ |
| 8 | $\frac{D(24)-D(21)}{24-21}$ | $D(21)$ |
| 9 | $\frac{D(27)-D(24)}{27-24}$ | $D(24)$ |
| 10 | $\frac{D(30)-D(27)}{30-27}$ | $D(27)$ |

*Figure 7: Hans' programming the camera's distance traveled over time using the car's average speed in each three-second interval*

15
$$\frac{D(30)-0}{30-0}$$
$= 0.878466666667$

16
$$\frac{D(30)-D(29.)}{30-29}$$
$= 2.437$

*Figure 8: Hans moving from* $\frac{D(30)}{30}$ *to* $\frac{D(30)-D(29)}{30-29}$

The interviewer then asked Hans how he might determine the speed at $t = 30$. Hans initially stated "$\frac{D(30)}{30}$ since it is the rate of change for 30… err… it's just a rate, not a rate of change." He later clarified that $\frac{D(30)}{30}$ "is a rate [and not a rate of change] because it remains a constant value of 0.878 like it would be the constant rate at 30." The interviewer then wrote $\frac{D(30)}{30}$ as $\frac{D(30)-0}{30-0}$ and asked what $\frac{D(30)-0}{30-0}$ would represent in this context. Hans identified that $\frac{D(30)-0}{30-0}$ would be the car's average speed between 0 and 30 seconds, and then quickly realized that $\frac{D(30)}{30}$ would not be the car's speed at 30 seconds. Afterward, he suggested we could determine the average speed between 29 and 30 seconds since "it's in the region that we want it in like 30, so it's close enough to the speed" (Figure 8). On the other tasks that prompted Hans to determine the speed at a given time value, Hans' responses indicated that he would determine the average rate of change over a small interval near the requested time (Figure 9).

What would you do if she asked for the speed at $t = 10$ ?

$\frac{D(10)-D(9.99)}{10-9.99}$ or $\frac{D(10.00001)-D(10)}{10.00001-10}$
or
$\frac{D(10.01)-D(9.9999)}{10.01-9.9999}$

Submit

If the director gave you a particular time that she wanted to know the speed for, describe what you would do to determine that for her

Finding a small interval near her requested time and then find the average rate of change in that interval.

Submit

*Figure 9: Hans' explanation of how to find a speed at a given input value*

Several aspects of Hans' responses are important to discuss.

1. Hans thought about $\frac{D(30)}{30}$ as a "rate not a rate of change." His expression, $\frac{D(30)}{30}$, suggested that a fraction with a singular distance divided by a singular time would determine a

"rate," and something of the form $\frac{y_2-y_1}{x_2-x_1}$ would determine a "rate of change" since the numerator and denominator represented a change in one quantity's value. It was likely through reconceiving $\frac{D(30)}{30}$ as being equivalent to $\frac{D(30)-D(0)}{30-0}$ (which he identified as the car's average speed between 0 and 30 seconds) that supported Hans in conceiving $\frac{D(30)}{30}$ as the car's average speed over 30 seconds.

2. Another possibility for his response of $\frac{D(30)}{30}$ as the car's speed at 30 seconds is that he considered that a rate was "distance over time," and so he used the distance at 30 seconds, $D(30)$ and divided it by the time value of 30 seconds. In either case, what supported Hans in shifting his conception of $\frac{D(30)}{30}$ was the prompting to explain what each portion of his expression represented. Additionally, writing an equivalent expression in the form that he was familiar with (e.g., recognizing that $\frac{D(30)-D(0)}{30-0}$ would represent an average speed) aided him in reflecting on the quantities he attempted to represent.

Like Scott, Hans also demonstrated an understanding of the arbitrariness of choosing how small an interval had to be when determining the speed at a given input value. When determining how he might determine the speed at $t = 10$, Hans initially wrote $\frac{D(10)-D(9.99)}{10-9.99}$ but also noted that he could choose other sized intervals like the subsequent ones he wrote in Figure 9.

## Discussion

The findings of this study present a possible alternative to how Calculus instructors can introduce the idea of instantaneous rate of change compared to the traditional sliding secant line methods used by popular textbooks (e.g., Stewart). In this study, both students benefitted from connecting their mathematical expressions with an applet to model their mathematics. Additionally, the applet provided a medium for the students to visualize how choosing small enough input intervals could be useful in approximating an instantaneous rate of change. Compared to the standard receding-secant line method, where the secant lines converge to one single point (in other words, a demonstration of approximating the instantaneous rate of change at only one particular value), the students in this study were engaged with developing an entire rate of change function as they constructed their piecewise linear functions. As a result, both students in this study demonstrated that they likely internalized this process of approximating an instantaneous rate of change since they described the process as involving an average rate of change over a small enough input interval.

While motion-based contexts should not be the only examples Calculus students should interact with, they can provide a tangible situation that students can associate their mathematical understandings with (Berry & Nyman, 2003). This study builds off the existing literature (Ely & Samuels, 2019) of alternative ways to introduce the idea of derivatives to support students in quantitative reasoning.

### Limitations and Future Directions

While the results of this study show promising results for alternative teaching methods, the results can only be known to be true for the students in the study. Future studies can continue to investigate how we can support students in reasoning covariationally in Calculus and how to implement these alternative methods in the classroom.

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