Ways of Reasoning About Inverse Functions

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Prior research on student understanding of inverse function has primarily focused on deficits in these understandings without explicitly addressing the mental actions and operations that researchers or instructors would like students to engage in. In this report, I present three ways of reasoning about inverse functions: a covariational approach, a mapping approach, and a set theoretic approach. I argue that all three entail productive ways of thinking about inverse functions, and that the coordination of the three can support students in reasoning about inverse functions in a variety of contexts.

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To intentionally design curriculum and plan instruction that targets the development of strong mathematical meanings on a particular topic, mathematics educators must first decide what specific mathematical meanings are most beneficial for students to construct (Steffe, 2007; Thompson, 2013). This includes a need to understand the mental operations and ways of thinking that are entailed in these desired ways of reasoning (Thompson, 2008; Thompson, 2013). Previous literature has primarily focused on student difficulty with inverse functions instead of providing a coherent description of how someone might reason productively about them. There is ample evidence that students often do not have coherent meanings for inverse function across contexts, but there are few detailed descriptions of what coherent meanings might entail. In this paper, I propose three ways of reasoning about inverse function and argue that all three are compatible with each other and useful for students, and that they should be targeted in teaching.

Literature Review

Although many researchers have studied student learning and understanding of function (e.g., Breidenbach et al., 1992; Carlson, 1998; Monk, 1992; Doorman et al., 2012) there has been less of a focus on inverse function in particular. Early work by Even (1990; 1992) examined the use of the metaphor of "undoing" to understand inverse function and several research groups have used characterizations of students' conception of function developed by Breidenbach et al. (1992) to extend to the context on inverse functions. Breidenbach and colleagues defined an action versus process conceptualization of function, where an action level of function involves the manipulation of objects (either physical or mental) and might allow a student to plug numbers into an algebraic expression to calculate a result but would limit the student to consider the result of such a calculation separately. A student with a process conception of function thinks about functions as processes that transform objects into other objects via some repeatable transformation but does not need to calculate any particular values to be able to conceptualize this process (Breidenbach et al., 1992). Vidakovic (1996) claimed that students with a process or object concept of function can develop concepts for inverse function. She argued that coordinating inverse function as the reverse of a function process, as the coordination of two function processes to get identity, and as an action of switching x and y and solving for y, is necessary for a deep conceptual understanding of inverse function but did not detail how a student might coordinate these different ways of thinking about inverse. Engelke and colleagues (2005) used the action versus process distinction and argued students who have developed a process view of function understand that this process can be reversed to obtain an inverse which would allow them to make sense of inverse function notation such as $f^{-1}(f(x)) = x$.

Oehrtman and colleagues (2008) discussed the implications of an action versus process conception of functions, and noted that an action view of functions makes it impossible to reason about functions dynamically because each input and output must be considered separately. Oehrtman et al. (2008) identified three distinct conceptions of the inverse of a function: as an algebra problem, as a geometry problem, and as the reversal of a process. They argued that students with an action view of function can understand inverse functions in the context of algebra (as a procedure to find the inverse of a given function) and the context of geometry (as a procedure to reflect a given graph over the line $y = x$), but do not understand why these procedures work or their connection to the ideas of composing or reversing functions. In contrast, students with a process conception of function can understand the inverse of a function to be the reversal of a process, which maps a set of outputs of the original function back to the associated inputs (Oehrtman et al., 2008). This way of reasoning requires students to be able to consider a function as a general process without having to coordinate individual input and output values as would be necessary for students with an action concept of function.

Paoletti and colleagues (2018) conducted task-based interviews that included contextualized and decontextualized inverse function tasks in analytic and graphical representations. Many participants tried to use the "switch and solve" procedure in contextualized contexts, which proved problematic because switching the variables led to a non-normative relationship between the two quantities. For example, when finding the inverse of a function that converts degrees Fahrenheit (F) to degrees Celsius (C), interchanging the variables F and C leads to a function that does not maintain the standard relationship between Fahrenheit and Celsius. Many of these participants concluded that their inverse was not meant to represent the same relationship as the original function or were unsure of how to interpret the inverse function that they found. Only one student demonstrated what the researchers inferred to be well connected meanings between analytic and graphical tasks in decontextualized and contextualized settings. This participant used "switching" techniques for all of the tasks, but she identified that she would need to change which quantity her variables referred to when switching in the tasks with quantitative context.

While most research on students' understanding of inverse functions has focused on the deficits and disconnected nature of this knowledge rather than on explicating potential ways that students might reason productively, Paoletti (2020) presented an example of a student successfully reorganizing her previously disconnected meanings for inverse function. The participant originally used switching techniques to solve tasks that did not support her in making sense of contextualized inverse function tasks. Throughout the course of the teaching experiment, the participant came to understand that a function and its inverse represent the same relationship in a contextualized situation, and that the switching and solving technique is used to maintain the convention that the quantity seen as the independent variable is represented on the x axis. Based on the student's actions, along with an analysis of research on covariational reasoning, Paoletti (2020) provided a detailed conceptual analysis of inverse functions that emphasized the invariance of the relationship between the two quantities related by a function and its inverse. This conceptual analysis will be discussed in more detail below.

More recently, Cook et al. (2023) conducted a conceptual analysis on the construct of inverse more generally (rather than specific to inverse functions), and proposed three ways of reasoning about inverse that can support students' understandings in a variety of algebraic contexts. Their first characterization is "inverse as undoing", where "inverse is viewed as a relationship between operations” and where “the purpose of the operation (or sequence of operations) in question is to undo the effect of the original operation(s)” (Cook et al., 2023, p. 757). They also argued that inverse can be thought of as “inverse as a manipulated element” in which students conceptualize an inverse as an element rather than an operation, and where this inverse element is “associated with a procedure by which a given element is manipulated into its inverse element”)” (Cook et al., 2023, p. 757). Lastly, they proposed “inverse as a coordination of the binary operation, identity, and set” (Cook et al., 2023, p. 757) where students view inverse as a relationship between a pair of elements such that their interaction yields the identity, with whatever binary operation is relevant in the given context. This conceptual analysis provides potential ways for students to see commonalities in the broader construct of inverse across mathematical contexts but does not provide explicit ways that students might reason about particular inverse functions.

Three Ways of Reasoning About Inverse Function

To develop the ways of understanding inverse functions presented here, I conducted a conceptual analysis. Conceptual analysis is a method grounded in a radical constructivist epistemology that involves a fine-grained examination of the concepts surrounding a particular topic (Thompson, 2008). Importantly, the analysis is not situated in the abstract but instead the researcher considers how an individual might think about the concepts. Conducting a conceptual analysis can be a way to develop answers to the question “What mental operations must be carried out to see the presented situation in the particular way one is seeing it?” (von Glasersfeld, 1995, p. 78). Thompson (2008) identified conceptual analysis as a fruitful technique for describing productive understandings for students to develop, and for examining the coherence of different ways of understanding a collection of ideas. To this end, I conducted a conceptual analysis on inverse functions to describe potential productive ways for students to reason about inverse functions, and to examine the coherence (or lack thereof) of these ways of reasoning. Based on my own mathematical understanding and my review of the literature on inverse functions, I have identified three distinct approaches to thinking about inverse function: a covariational approach, a mapping approach, and a set theoretic approach.

A Covariational Approach to Inverse Function

Covariational reasoning, as defined by Carlson et al. (2002), involves “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354). This way of reasoning has been shown to be powerful for helping students reason about dynamic situations (Moore & Carlson, 2012), exponential growth (Ellis et al., 2012), calculus (Thompson et al., 2013) and trigonometry (Moore, 2014). Thompson and Carlson (2017) proposed a covariational meaning for the concept of function:

> A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person’s conception, every value of one quantity determines exactly one value of the other. (p. 444).

This construct was extended to define the idea of a “covariational relation” (Paoletti, 2020), in which a student has conceptualized two quantities varying together where one quantity varies first and the other quantity changes along with the first. This can then become a covariational function once the student constructs the property that each value of the first quantity corresponds to exactly one value of the second (Paoletti, 2020). A student who reasons about functions from a covariational perspective can imagine the first quantity changing and knows that the second quantity will also change correspondingly, without having to think about or calculate actual values of either quantity. For example, a student might construct a covariational relation between height and volume when considering water being poured into a pool, where they consider the height of the water first and then the associated volume at that moment. If the student realizes that for any height value, there is a unique corresponding volume, they have then constructed a covariational function of volume with respect to the height of the water.

If a student then were to consider the volume changing first instead of the height, they would have constructed an inverse covariational relation (Paoletti, 2020). If they realized that for each value of the volume there is exactly one associated value of the height of the water, they would have constructed an inverse covariational function (Paoletti, 2020). Once a student has constructed both a covariational function and an inverse covariational function to describe the same two quantities varying, they can see that both the original function and its inverse represent the same relationship between the quantities, and the only aspect that changes is which quantity is considered to be varying first, with variation in the second dependent on the variation of the first. This does not imply an actual causal relationship between the two covarying quantities, but instead refers to the fact that the amount of variation in one quantity that is considered is dependent on the amount of variation considered in the other. From this perspective, there is no need to consider the inverse relationship as a separate entity from the original relationship, it is simply the same relationship viewed from a different perspective.

In a quantitative context in which one quantity is not uniquely determined by the other, a student can still conceive of an inverse covariational relation. For example, imagine that the student had instead considered the volume of the water in the pool as time varied, with the pool first being drained and then refilled. In this case, each value of the volume of water in the pool would correspond to (at least) two different times, one as the pool is emptied and one as it is filled. The student can still consider volume varying first with the associated time values varying with it, despite the lack of uniqueness. For example, the student can imagine that as the volume of water in the pool increases, one of the corresponding time values also increases (becoming closer to the time when the pool is completely refilled) while the other decreases (becoming closer to the time when the pool started to drain). The mental actions that a student engages in when constructing an inverse relation and an inverse function are the same from this perspective, and an inverse function can be seen as just a type of relation that happens to fulfill the uniqueness property. This flexibility in imagining inverse covariational relations that are not necessarily functions is in direct contrast to the way that inverse functions are typically taught, in which students learn that functions are not invertible if they are not injective. This may lead students to believe that it is impossible to construct an inverse relation in these situations, or that these inverse relations would have no use.

A Mapping Approach to Inverse Function

Traditionally, functions are defined as mappings that take each element of a function's domain to exactly one element of the function's range. To construct the idea of function from a mapping perspective, a student must conceptualize two arbitrary sets with some sort of rule that takes elements from one set as inputs and gives elements of the other set as outputs, and where each possible input maps only to one output. To construct the idea of an inverse function, a student must then imagine another function that maps in the opposite direction, taking the original function's output values to the associated input values. To be able to define this map that reliably takes an output value to its associated input, the original function must satisfy the property that each input maps to a unique output value. A student must see that if this were not the case, there would be no way to determine which input value the inverse function should map the output value back to, and thus the injectivity of the original function becomes a requirement to define an inverse function. The student can then imagine elements from the function's domain being mapped to elements in the range by the function, and then these elements of the range being mapped back to the original elements in the domain when the inverse function is applied. The opposite order of applying these mappings will yield the same result, where elements of the range of the original function are mapped to elements of its domain by the inverse function, and then applying the original function will map these elements back to their "starting point".

With a covariation approach to function, the student imagines changes in each quantity occurring simultaneously and attends to these changes in relation to each other (e.g., as quantity A increases, quantity B decreases). With a mapping approach to function, the student instead imagines change from one set to the other (e.g., an element, x, being "transformed" by the function to an element, y). Both approaches have an imagined dynamic element, but the types of change that are emphasized are fundamentally different. In contexts where students are reasoning with actual quantities, often a transformation approach to understanding function is incompatible (at least from the perspective of the researcher) because one quantity cannot "transform" into the other. Recall the example above where the volume of water in a filling pool is seen as a function of time elapsed. This function cannot transform time into volume, but instead simply provides a relationship between the values of the two quantities. Additionally, in contrast to the covariation approach where the inverse is seen as maintaining the same relationship as the original function but from a different perspective, the mapping perspective instead views the inverse as a separate function in its own right. Instead of the emphasis being on the sameness of the relationship defined by the function, the emphasis is instead on the relationship between the function and its inverse, namely that the inverse undoes the action of the original function.

A Set Theoretic Approach to Inverse Function

Relations can also be defined from a set theoretic perspective, as a set of ordered pairs (a, b) where the first element in the pairs are elements of a set A and the second element in the pairs are elements of a set B. The set A is defined to be the relation's domain, and the set B is the relation's range. In contrast to the covariation and mapping approaches, there is no mental imagery of change or variation, only of static ordered pairs being elements of a relation. From this perspective, in order for a relation to be a function, the property must hold that if (a, b) and (a, c) are elements of the relation then b and c are the same element of the range. The inverse relation would then be the set of ordered pairs (b, a) obtained from reversing the order of the pairs in the original relation and thus interchanging the domain and range. This inverse relation would be a function if it satisfies the above property. Using this definition to think about functions and inverse functions leads to an understanding of inverse as a switching of coordinate points. This is useful for mathematical settings that do not involve continuity or that are in contexts outside of the real numbers but does not support the use of quantitative reasoning to imagine "real life" situations, particularly those in which images of (co)variation are critical. If functions are thought of solely as sets of ordered pairs, then there is not a natural way to envision how one quantity might change in relation to another.

Despite these limitations of a solely set-theoretic approach, this approach is in no way incompatible with a covariation or mapping perspective. All three perspectives foreground pairs of values, but the set theoretic approach does not necessitate the consideration of variation between the pairs of values or a transformation from one value to another. From a covariation perspective in which two quantities are varying simultaneously, one must keep in mind that for any given value of one quantity there is an associated value of the other quantity, and these pairs of values can be seen as the ordered pairs considered from the set theoretic perspective. Similarly, the mapping perspective emphasizes that values from the domain are mapped to values in the range of a function, and each input/output pair can be seen as the ordered pairs from the set theoretic perspective. The set theoretic approach assumes the existence of these pairs of values but does not support someone in producing the pairs. In order to produce the pairs, one must use either a covariation or mapping approach to function to determine the value of one quantity for a given value of another quantity.

Examples

Each of the three ways of reasoning described above provide useful ways for students to reason about inverse functions, and it is the goal-oriented activity that a student engages in that dictates which way of thinking is best suited for the situation. For example, consider a quantitative context where two quantities covary and in which it can be useful to think of either quantity as varying first. A covariation perspective can allow a student to reason flexibly about either quantity varying first, with the persistent realization that both ways of thinking about the context maintain the same relationship between the quantities. A student reasoning in this way can view the graph of a function as also representing the graph of the inverse by considering the quantity on the vertical axis as varying first instead of the standard view of values on the horizontal axis representing the independent variable. This can allow them to reason about the inverse relationship using only a graph of the original function without needing to construct a separate graph of the inverse. In this context, a mapping perspective is somewhat less natural, and in order to reason about the inverse of a given function a student would be more likely to have to construct an equation or graph for the inverse first.

However, in contexts involving composition of functions, thinking of inverse from the mapping perspective is more productive. From a covariation perspective there is usually no motivation to consider the inverse relationship as a separate function from the original, and so composition is not obvious. From the mapping perspective the fact that a function composed with its inverse should yield the identity function becomes clear. If the inverse of a function is a function that maps the outputs of a function back to the original inputs, then performing one mapping and then the other must always result in whatever the original input was. When students are asked to produce a graph of the inverse function based on the graph of a given function, set theoretic reasoning is often most efficient. The original graph of the function consists of the ordered pairs of that function, and thus a graph of the inverse function must consist of these same pairs but in the opposite order. This reasoning can allow a student to bypass the cognitive effort of the analogous reasoning from a mapping perspective, where a student might reason that if a point on the original graph (x, y) means that the function maps x to y, then the inverse function must map y back to x and thus the point (y, x) will be on the graph of the inverse.

Despite the varying degrees of utility of each way of reasoning in different contexts, the three ways of understanding inverse function are in no way incompatible. Consider the example of the relationship between temperature measured in degrees Fahrenheit (F) and Celsius (C): $C=5/9(F-32)$. From a covariation perspective, a student can reason that this equation gives a relationship between the measures of temperature in both units and can see a graph of this relationship as providing information about the relationship viewed with either quantity varying first. At the same time, that same student can know that, as written, the function takes values of temperature measured in Fahrenheit and maps them to the appropriate value measured in Celsius, but that they can also find a function that takes values of temperature in Celsius and maps the opposite way back to the appropriate value in Fahrenheit. Along with these ways of reasoning about the situation, the student can also understand that the original function can be thought of as ordered pairs (*F,C*) that are solutions to the equation above, and that the inverse relationship can be seen as the pairs in the reverse order (*C, F*). Depending on the student's goal when reasoning about this context, any of these three meanings could be drawn on productively.

Conclusion

The ways of reasoning described above provide a finer grained description of the mental actions involved in reasoning about inverse functions. Previous work has provided hierarchical frameworks for the development of an understanding of inverse functions (e.g., the action vs. process conceptions of inverse functions), but the details of what an understanding of inverse functions entails were underdeveloped. The action versus process dichotomy does not capture differences between a student reasoning from a set theoretic, mapping, or covariational perspective. While the description of the covariation perspective on inverse functions is not new (Paoletti, 2020), the juxtaposition of this perspective with the mapping and set theoretic perspectives offers a broader view of productive meanings for inverse function. In addition, the conceptions of inverse functions identified by Oehrtman et al. (2008) separated conceptions of inverse mostly by the procedures or products of reasoning associated with them (i.e., reflecting over a line to graph, switching variables and solving, or understanding inverse as the reversal of a process), without specifying the associated mental operations behind the three conceptions. This categorization gives us three ways that students must be able to engage with inverse functions but is of limited use in specifying how we would like students to think about inverse functions. The conceptual analysis conducted by Cook et al. (2023) also identified three ways of reasoning about inverse, but in the more general context of inverse not specific to inverse functions. I argue that, while there is overlap between the characterization presented in this paper and that provided by Cook et al. (2023), there are ways of reasoning that are important for productive engagement with inverse functions that are not captured in broader reasoning about inverse. For example, a covariational approach to inverse function relies on continuity, and while it is a useful way of reasoning for students in these contexts, there is not necessarily an analogous form of reasoning when considering inverse elsewhere in mathematics. In turn, the characterization of inverse function meanings provided here does not capture meanings about inverse functions as an instantiation of the larger idea of inverse, and I have not attended to how students understand functions and their inverses as elements of a set.

A significant limitation of the characterization of inverse function reasoning presented here is that it is a first order model (Thompson, 2000) that it is based on my own mathematical meanings. It is informed by the extant literature base on student understanding of inverse functions and by my interactions with students but has yet to be studied empirically. Future research should examine how these ways of reasoning emerge in practice and explore ways in which the coordination of these different ways of reasoning might support engagement with inverse function tasks. If these proposed ways of reasoning about inverse function maintain their viability after empirical research with students, then further research could begin to explore the development of these ways of reasoning. Reflected abstraction (Piaget, 2001) can help to connect previously disjoint mathematical meanings into one coherent scheme and brings these connections to the level of conscious awareness. Therefore, I hypothesize that reflected abstraction is a potential way for students to integrate these distinct ways of understanding inverse function, and that supporting students in engaging in reflected abstraction may encourage coherence in their meanings.

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