

College Students' Conceptualizations of Symbolic Algebraic Properties

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Here we explore how college students across a wide range of courses may conceptualize symbolic algebraic properties. We draw on the theory of Grundvorstellungen (GVs) to analyze how learner conceptions may or may not align with instructional goals. In analyzing interviews, several categories of conceptions (descriptive GVs) emerged that may help us to better understand how students conceptualize symbolic properties during instruction.

Keywords: Algebraic properties, Syntactic reasoning, Equivalence, Algebraic transformation.

Mathematical properties are critical to justifying symbolic transformation, especially in algebra and domains that rely on algebraic representation. However, learners often use properties in ways that are not mathematically valid (e.g., Hoch & Dreyfus, 2004; Mok, 2010) and instruction may not address the use of symbolic properties (and their role in justifying transformation) explicitly enough (e.g., Barnett & Ding, 2019). Here we focus on learners' ability to identify parallel syntactic structure between symbolic properties and symbolic algebraic representations, and we explore how this may relate to learners' conceptions.

Properties and Forms

Because here the focus is on how properties are used to transform symbolic representations, we define a *symbolic property* as any mathematical statement that can be used to transform a symbolic object into an equivalent one with a different form. Two examples are: 1) the definition of negative exponents, e.g.: $x^{-n} = \frac{1}{x^n}$ for $x \neq 0$; and 2) this statement about equivalent equations: $A \cdot B = C \leftrightarrow A = \frac{C}{B}$ (when $B \neq 0$). The key feature of this definition is that 1) could be used to replace an expression of the form x^{-n} with one of the form $\frac{1}{x^n}$ (or vice versa), and 2) could be used to replace an equation of the form $A \cdot B = C$ with one of the form $A = \frac{C}{B}$ whenever $B \neq 0$ (or vice versa). Symbolic properties are made up of two sides, each of which can be viewed as a separate object, which are often referred to colloquially during instruction as “forms” (e.g., the “forms” x^{-n} , $\frac{1}{x^n}$, $A \cdot B = C$, and $A = \frac{C}{B}$ above).

Relatively little is known about learners' conceptions of symbolic properties. Existing research has focused on classifying errors that learners make when using properties to compute or transform (e.g., Hoch & Dreyfus, 2004; Mok, 2010); on learners' justifications for why properties are true; or on learners' ability to derive properties from arithmetic patterns (e.g., Hunter et al., 2022). Schüler-Meyer (2017) has investigated learners' structure sense for the distributive property (e.g., Schüler-Meyer, 2017), there is a dearth of research looking at this for symbolic properties more generally. Given how critical using and understanding symbolic properties and forms is for transforming symbolic representations (Kieran, 2011), it is essential that more research investigate how students conceptualize symbolic properties more generally, to address this gap in the research literature.

Theoretical Framework

Grundvorstellungen

We use prescriptive and descriptive *Grundvorstellungen* (GVs) (or “fundamental conceptions”) to frame this research. *Prescriptive* GVs describe conceptions that are the goal(s) of instruction (vom Hofe, 1995); *descriptive* GVs describe actual conceptions that learners hold, which may or may not reflect prescriptive GVs. Comparing these two types of GVs can then be used to guide curriculum or instruction (Greefrath et al., 2016). Both descriptive and prescriptive GVs are intended to evolve with research over time. Further, one concept may have multiple GVs and vice versa. First we describe two *prescriptive* GVs for symbolic properties (Figure 1). After analyzing student interviews, we will present some descriptive GVs in the Results section.

Equivalence-Preserving GV	Symbolic properties describe a method for replacing one symbolic representation with another equivalent one, based on a context-dependent pre-existing definition of equivalence (e.g., insertion equivalence; Prediger & Zwetzscher, 2013)
Mapping GV	For equivalence to be preserved, the following criteria must be met: The form on one side of the symbolic property must be mapped bijectively to the symbolic representation so that: 1) A unified subexpression is mapped to each variable in the form; 2) All other symbols are mapped to notation in the form with the same syntactic meaning (e.g., different notations for multiplication can be mapped to one another).

Figure 1: Two Related Prescriptive GVs for Symbolic Properties

Operational vs. Structural Conceptions and Extracted vs. Stipulated Definitions

In constructing models of learners’ descriptive conceptions, we were also influenced by research on operational vs. structural conceptions (Sfard, 1992) and extracted vs. stipulated definitions (Edwards & Ward, 2004). A learner with an *operational* conception views properties as a process of computation, while a learner with a *structural* conception views them as abstract objects (e.g., canonical representations of particular algebraic structures). Sometimes learners treat something as an object that is not the reification of any process, and this is called a *pseud structural* conception (Sfard, 1992, p. 75). The operational/structural distinction is related to the prescriptive Mapping GV of Symbolic Properties, which focuses on conceptualizing forms within a property structurally as objects.

Extracted definitions emerge organically from observed usage of a term (e.g., when a learner extracts meanings for a property based on how it was used during computation in instruction). In contrast, *stipulated* definitions are explicitly stated—to determine if something fits the definition, one must consult the definition directly (Edwards & Ward, 2004). The extracted/stipulated distinction is related to the Equivalence-Preserving GV of Symbolic Properties, as a core stipulated part of the properties definition is that they preserve equivalence (in addition, the type of equivalence that is preserved must be based on a stipulated definition).

Methods

This project is based on 102 cognitive interviews that were conducted with college students in the US in 18 courses ranging from elementary to linear algebra on items from the Algebra Concept Inventory (Wladis et al., 2018); courses included both STEM and non-STEM courses. Students interviewed were diverse in terms of gender, race/ethnicity, national origin, and English language learner status. Thematic analysis (Braun & Clarke, 2006) was combined with an initial

theoretical stance focused on noticing both extracted vs. stipulated definitions (Edwards & Ward, 2004) and operational vs. structural (Sfard, 1992) conceptions, as well as the extent to which learners provided potential evidence of Equivalence-Preserving or Mapping GVs. This allowed for the resulting coding framework of learners' descriptive GVs of symbolic properties to contain both emergent and confirmatory aspects.

Results and Discussion

Analysis of cognitive interviews resulted in a framework of learners' *descriptive* GVs of symbolic properties (Figure 2). Here operational vs. structural conception categories relate to how closely learners' GVs align with a Mapping GV and extracted vs. stipulated definition categories relate to how closely they align with an Equivalence-Preserving GV (Figure 1).

	Extracted Definition	Stipulated Definition
Operational Conception of Properties	Pseudo-process GV: Learners see properties as a cue to a computational process, and their approaches are extracted from prior experience rather than based on stipulated definitions. They often draw on surface structure rather than syntactic meaning. For example, learners may conceptualize the distributive property as an instruction to "take what is on the outside of the parentheses and put it next to each thing on the inside", regardless of the specific operations involved.	Process GV: Learners see properties as a cue to a computational process, but attend to syntactical meanings and/or equivalence as a justification (e.g., checking for appropriate operations in the expression; checking that original and resulting expressions are insertionally equivalent). However, they may struggle to conceptualize properties as objects to which structures in the expression or equation can be mapped one-to-one, and thus may have difficulty generalizing the use of properties to more syntactically complex symbolic representations.
Structural Conception of Properties	Pseudo-object GV: Learners conceptualize a property as something that requires mapping to the specific forms in the property, but the mapping is still somewhat ill-defined and/or based on extracted notions, such as what "looks right".	Object GV: Learners conceptualize the property as an object, such as a canonical form, to which the specific mathematical object (i.e., expression, equation, etc.) must be mapped one-to-one, in such a way that preserves syntactic meaning. They recognize that it is these criteria that preserve equivalence.

Figure 2: Framework to Categorize Descriptive GVs for Symbolic Properties

We illustrate the framework by presenting some examples of student work.

Operational Conceptions

Many learners appeared to draw on operational conceptions of symbolic properties. First we consider a student, Iota, who was enrolled in an introductory statistics course (elementary algebra was a pre-requisite), who was given seven questions with the following form:

- Q6: Which of the following could result from using the **distributive property** to rewrite the expression $(x + 2)(3x + 7)$?
- $x + 2 \cdot 3x + 7$
 - $x \cdot 3x + 2 \cdot 7$
 - $x + 2 \cdot 3x + 2 \cdot 7$
 - $(x + 2) \cdot 3x + (x + 2) \cdot 7$
 - None of the above.
 - I don't know the distributive property.

Figure 3: Task discussed with Iota during the interview

Other versions of this item used the following expressions: Q1: $(2x + 1)2$; Q2: $x - (2x + 1)$; Q3: $2(2x \div 1)$; Q4: $2(x \cdot y)$; Q5: $(2x + 1)^2$; and Q7: $2(xy)$. Iota stated that the distributive property could be used to rewrite the expression for each of these. They correctly chose equivalent expressions that could be the result of the distributive property for Q1 ($2x \cdot 2 + 1 \cdot 2$), Q2 ($x - 2x - 1$), and Q6 $(x + 2)(3x + 7)$ (option D). However, they also incorrectly chose “results” of the distributive property for Q3 ($2 \cdot 2x \div 2 \cdot 1$), Q4 ($2x \cdot 2y$), Q5 ($(2x)^2 + 1^2$) and Q7 ($2x2y$) that suggest that they may conceptualize the distributive property as an instruction to do something like “take what is outside the brackets and apply it to each ‘thing’ inside the brackets”. At the same time, Iota’s is able to conceptualize $(x + 2)$ as a unified subexpression within $(x + 2)(3x + 7)$ that could be “distributed” to each term in $3x + 7$, which suggests that Iota is able to think structurally in key ways. Iota explains:

Because obviously two can distribute [makes motion with fingers as though moving two from left to right twice] with the one in parentheses. So two in the front can distribute to $2x$ multiply by $2y$. So it's gonna be $2x$ multiply by $2y$ [repeats distributive motion with fingers]—that's the result.

Here Iota focuses solely on describing a computational process based on surface similarities, without considering the mathematical validity of that computation, consistent with a pseudo-process GV. We see further evidence of this later in the interview:

Interviewer: What is the distributive property?

Iota: Distribute property is like that you can use the main number or main groups to distribute to each of another number or another groups.

Interviewer: So is that like here [highlighting $(x + 2)$ in Q6], is $x + 2$ the main number?

Iota: It's a main group. Yes.

Interviewer: And then you apply that to each of the ones [motions to $3x$ and 7 in Q6]?

Iota: Yes.

Interviewer: Okay. So, I noticed that this one [highlights $+$ in expression $(3x + 7)$ in Q6] has a plus sign in between them. Is the distributive property only for the plus sign or could it also be subtraction? Could it be multiplication or division?

Iota: So, yeah, it could be subtraction, multiplication ... Could be any sign, but when you calculate, when you are doing it, you have to do with that own sign.

Here Iota provides further evidence that they are viewing the distributive property as an extracted process here, where whatever is outside the brackets is multiplied by each “group” inside the brackets, preserving the original operation between the multiple “groups” inside the brackets. However, when Iota was interviewed about Q7 ($2(xy)$), they appear to shift to a process view, checking for mathematical validity of the transformation results by checking insertion equivalence through arithmetic computation:

Iota: Sometimes when I see these kind of questions, at first I may think its right answer is A ($2x2y$), but what I normally do is I double check the answer. So, I create some equations and I double check it, it's incorrect. So, for this case, I create like x is 3. Okay, let me type it now, y is 2.

$$x=3, y=2$$

$$2(3 \cdot 2) = 2 \cdot 6 = 12$$

$$2 \cdot 3 \cdot 2 \cdot 2 = 24$$

I think it's wrong. So, I say no... I don't know why, but this is very tricky question for me ... So, x and y multiply each other should be done before multiply[ing] the one outside....I don't know, it's not look like a distributive property for me. It looks like the way to

calculate is you do the xy first because in parentheses, and after you get the result of xy you do with the number 2. So, I don't think this one is like a distributive property ...to be honest, I don't know why. I don't think it's A, but I just feel it's not.

Interviewer: This strategy that you were doing, replacing x and y with numbers and seeing if they were the same: if you did that for number 6, for example, would get the same answers?

Iota: Oh, that's a good question. I don't ... Yeah. Right. I don't know ... I didn't ... I didn't try. But ... I mean, I'm just, I'm looking at it right now. Yeah, it should be the same. Because it should be only one value. Mm-hmm.

In this excerpt there is evidence of both process and pseudo-process GVs. Iota now shows evidence of the prescriptive Equivalence-Preserving GV, because they substitute numbers to check whether the result of their distributive property computation in Q7 is insertionally equivalent to the original expression, at least for one value. When it is not, they then question their use of the distributive property to replace $2(xy)$ with $2x2y$, providing evidence of a process GV. However, their explanation still draws on extracted meanings and some pseudo-process GVs: they several times mention “feeling” that the distributive property is not correct or whether an expression “looks like” the distributive property should be used. In the other six similar distributive property items, they do not use a process GV; however, when the interviewer asks them directly whether this checking process should work for those also, Iota then draws on their knowledge of the distributive property as an equivalence-preserving transformation to recognize that this is also relevant for the other expressions. Whether the pseudo-process or process GV was cued for Iota appears to be linked to the way that different expressions “look”, which may be important to keep in mind when designing curriculum and instruction. It may be that instruction and tasks that focus more on checking and justifying calculation as well as linking the equivalence-preserving GV to calculation procedures, especially for a diverse problem space with many different forms, may be critical for learners like Iota.

Pseudo-object GV

The next interview was conducted with an elementary algebra student, whom we call Eta. They were asked to interpret whether $(2x + 1)(3x - 5)$ could be viewed as equal to the form $(a + b)c$.

Consider $(2x + 1)(3x - 5)$ in its current form (don't rewrite it or do anything to it). Is there any part of $(2x + 1)(3x - 5)$ which could be equal to $(a + b)c$ if we pick the right expressions to represent a , b , and c ?

- a. No
- ☒ b. Yes, if $c = 3x$
- c. Yes, if $c = 1$
- d. Yes, if $c = 3x - 5$
- e. Yes, if $c = 3$

Figure 4: Eta considering whether $(2x + 1)(3x - 5)$ can be seen as having the form $(a + b)c$

Eta: $2x$ could be a then the one would be b , then the c would be $3x$... if c is equal to $3x$ then it would make sense.... I'm just doing it by order by the first number, second number, third number. Maybe that's not the best way, but that's what I was doing.

Interviewer: What's being multiplied in each case [pointing to the expression]?

Eta: Two is being multiplied by three. Two is also being multiplied by the negative five. The same thing for the one, the one is being multiplied by three and then the one is also being

multiplied by the negative five.

Here Eta focuses on mapping subexpressions to variables in the form “in order”, which reflects a pseudo-object GV: they map the “first subexpression” to the first variable, etc., without attending to grammatical meaning of the syntax. When considering $(2x + 5)(3x - 5)$ and mapping subexpressions to the form $(a + b)c$, Eta appears not to “see” the second set of brackets around $3x - 5$ initially (or not recognize them as a grouping symbol); but when further questioned by the interviewer, Eta explains that each term in $(3x - 5)$ will eventually be multiplied by each term in $(2x + 1)$. This provides evidence that Eta’s pseudo-object GV of symbolic properties is likely not caused by a failure to recognize the syntactic role of the second set of brackets. Rather, Eta appears not to focus on the existing syntactic meaning of expressions when mapping that expression to a form. Eta appears to conflate the existing syntactic meaning of $(2x + 1)(3x - 5)$ with the result of expansion, perhaps literally conceptualizing $(2x + 1)(3x - 5)$ as having the syntactic meaning $2 \cdot 3 \cdot x^2 + 2 \cdot -5 + 1 \cdot 3 \cdot x + 1 \cdot 5$. However, while these two expressions are equivalent, they do not have the same syntactic meaning, and conflating the syntactic meaning of the first expression with the second one appears to obscure the structure needed to map this expression to the form $(a + b)c$. Thus, Eta’s computational view of syntactic structure may be impacting their GV of symbolic properties. Instruction that more explicitly highlights the differences in syntactic structure of different expressions and links this explicitly to form mapping, may better prepare Eta (and learners like them) to draw on their existing knowledge of syntax, symbolic structures, and forms as objects. Future research is needed to explore this possibility.

Object GV

We now consider an interview with an elementary algebra student whom we call Theta, who was asked to interpret whether $\frac{2x^2(y-1)}{2}$ could be viewed as equal to the form $\frac{(ab)}{c}$ (where $c \neq 0$).

Consider $\frac{2x^2(y-1)}{2}$ in its current form (don't rewrite it or do anything to it). Could $\frac{2x^2(y-1)}{2}$ be equal to $\frac{ab}{c}$ if we pick the right expressions to represent a , b , and c ?

- a. No
- b. Yes, if $b = x$
- c. Yes, if $b = x^2$
- ☒ d. Yes, if $b = y - 1$

D

Figure 5: Theta mapping a multi-term expression to a variable in a form

Theta: I felt like D was the best option because looking at a and b over c the first equation fit that like a could be $2x^2$ squared and b could be $y - 1$ and c could be 2.

Interviewer: Did the parentheses impact your decision?

Theta: Yes.

Interviewer: How?

Theta: Because I saw that the $y - 1$, I saw it as separate from $2x^2$. And I know that looking at the second one that a and b in order for them to be multiplied they would most likely have to have parentheses around them. And I saw $y - 1$ in parentheses so I just ... looking at them all as substitutes, as soon as I saw a and b over c like I was just putting in my head okay, $2x^2$ squared is a , $y - 1$ is b , and the two is equal to c .

In this excerpt Theta provides evidence of an object GV. They identify mathematically valid subexpressions in $\frac{2x^2(y-1)}{2}$ and identify which of these should map to which variable in the form to preserve the structure. Later the interviewer asked Theta to identify different syntactic structures in the expression, and Theta was immediately able to do so correctly. Here Theta also appears to conceptualize brackets from an object view (as a grouping mechanism rather than a cue to a procedure [see Wladis et al, 2022a]) because they “separate” the $2x^2$ and $y - 1$. Because Theta’s object view of syntactic structure is critical to their identifying the subexpression structures that will create a syntactic-structure-preserving one-to-one mapping from $\frac{(2x^2)(y-1)}{2}$ to the form $\frac{ab}{c}$ illustrates how this object view of syntactic structure may be a critical precursor to having an object view of symbolic properties. Theta also specifically mentions substitution when describing how subexpressions relate to the properties form: thus, Theta’s notions of substitution and substitution equivalence may be related to their symbolic properties conceptions (see Wladis et al., 2022b). Theta’s explanations here are substantially more structural than most other students in the sample (including those in a wide variety of course levels). While this evidence is not causal, Theta’s responses indicate that some elementary algebra students are capable of reasoning structurally about symbolic properties. Theta was part of an intervention that was designed to teach students the prescriptive GVs presented here (as well as others related to syntactic structure and equivalence). This may have influenced their GV formation; ongoing research is underway to explore this possibility. But regardless of whether this particular intervention played a role, Theta’s responses show how elementary algebra students are capable of thinking structurally.

Conclusion

These different vignettes illustrate how conceptualizing student thinking around symbolic properties using the framework in Figure 2 may be productive for understanding the reasons that students work with symbolic properties in particular productive or non-productive ways. One interesting pattern across all three vignettes is that each of the learners shows evidence of potentially productive prior knowledge, however, the extent to which they were able to use this prior knowledge productively in the context of symbolic properties varied quite a bit. We also saw here how student conceptions of symbolic properties are also intricately related to their conceptions of symbolic structure and equivalence, and thus some conceptions of these related concepts may be essential precursors to student conceptions of symbolic properties (Wladis et al., 2023). More research is needed to explore the relationship among these various conceptions, as well as what factors enable or disable students from productively drawing on prior knowledge when working with symbolic properties. We continue to investigate these relationships in ongoing research, and hope that others will as well.

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