Investigating How Students Try to Solve Real Analysis Problems

Brandon Watson
Texas State University

This study observed real analysis students attempting the homework assigned by their instructor to identify how they used resources, specifically example proofs, to complete their homework. The data consisted of seven students in total over three instructors. Overall, the study found that students tended to adopt a recurrent style of how they used their notes, or did not use their notes, to advance their solutions. However, there were some instances where the students would deviate from their style which could be linked to the type of problem as well as how the instructor taught that topic. This study contributes insight into how real analysis students may use their resources for the tasks assigned by the instructor.

Keywords: Problem situation image, Tentative solution start, Resource framework

Introduction

In Real Analysis, instructors introduce definitions and theorems which are part of the material that the student needs to know. Theorems and other statements come tied to proofs that are both material the students need to know as well as something the students must learn to write. The proofs that the instructor demonstrates in-class, thus, serve a dual purpose. One role of the proof is to prove the theorem/statement the instructor has introduced. The second role of the proof is to serve as a demonstration of the process of proving to the students. Research has already suggested that lecturers tend to recommend their notes as the primary resource for students (Ni She, Bhaird, Fhloinn, & O'Shea, 2017). It stands to reason that a skill the students should develop is to be able to reference the proofs they do in class as examples to inform themselves on how to write the proofs that they are asked to do for homework.

Studies have shown that students do tend to use curriculum materials and class notes as a primary resource when working on class material (Anastasakis, Robinson, & Lerman, 2017; Ni She, Bhaird, Fhloinn, & O'Shea, 2017; Pepin & Kock, 2021). Albeit, the students may leverage certain resources more heavily when with certain goals such as high grades or simply passing in mind (Anastasakis, Robinson, & Lerman, 2017). This study wanted to dive deeper into *how* students used their resources. As such, this study documented students working on homework tasks and found that students tended to have a style on how to use their class notes to complete homework tasks. Interestingly, certain homework tasks tended to deviate students away from their usual style. This seemed to occur due to certain features of the task and how the instructor presented related proofs. Building on an existing framework, this study extends the literature on how students use resources and worked examples in the context of proof-based mathematics courses.

Research Questions

1) When and how do real analysis students leverage the examples presented in-class to complete their homework tasks?
2) What factors influence the reason that real analysis students use their notes to complete their homework tasks?

Resource use has been investigated by a number of researchers (Galbraith et al., 2000; Guedet & Pepin, 2018; Muir, 2014; Radovic & Passey, 2016). However, research on resource use remains limited for a proof-based mathematics context. This study focuses on that context in an effort to broaden our understanding of how students use their instructor-provided resources for the problems they are assigned for homework.

Firstly, it is important to describe how a student comes about making a solution attempt for a homework problem. While working on homework, students may have different problem situation images associated with the topics in the question as well as the task assigned to them (Selden et al., 1999). Students typically make an initial judgement of the difficulty of the task and assesses whether they need to reference their notes or if they think they know how to do it. This initial assessment leads the student to their initial tentative solution start (Selden et al., 1999). For this study, it was important to note the instances of when students did not feel the need to reference their notes or note what they referenced when they did reference their notes. This helped define *when* students would use their notes by looking both at the task itself as well as how a student initially responded to the task.

As for *how* students used their resources, I adapted the technology framework developed by Galbraith, Goos, Renshaw, & Geigher (2000) into a Resource Use Framework. I conjectured the use between resources and technology can be similarly classified since in both cases the object being used by the student is still a tool for completing a task. As such, there are four categories that students could be classified for how they used their resource:

Table 1: Resource Framework

Resource as Master	A student copies a proof or makes every decision based on their notes without a way to check the adequacy of their decisions
Resource as Partner	A student has a tentative strategy but searches their notes to either find a new strategy or justify their own, or the student finds a relevant example but needs assistance to adapt it into a solution
Resource as Servant	A student either has a strategy and uses their notes to complete their strategy, or a student finds a relevant example and is able to fully adapt it into a solution without assistance
Resource as Extension of Self	A student knows exactly what they need from their notes so the notes act more as an extension of memory rather than a tool

Data Collection

Data was collected from three Real Analysis classes over two semesters each of which had different instructors from a large, public university in the United States. I observed the instruction and took field notes. The field notes consisted of what was written on the board, the comments the instructor added, and questions or remarks the students made. The field notes provided a detailed reference for the information the students received during class and what their resource of written notes could possibly resemble when they are doing their homework.

I recruited students from the aforementioned classes. In total, seven students participated. The student interviews consisted of asking students to attempt problems from their assigned homework. Priority was given to problems that I could connect to parts of the instruction that

occurred in class. This provided an opportunity to observe whether the student needed the resource from class or could identify the same resources I had if they did look into their notes. Students were videoed as they attempted to solve each problem. The video provided both verbal and visual evidence of what the student was doing at the time. The overall structure of the interview had three parts per question. Firstly, I let the student attempt the problem independently, noting whether or not they used any resources. If they formed a solution independently, I moved on to the next task. If not, I would ask them if they knew of anything from class that could help. Sometimes the student would find something and other times they would not find anything. If the student did not find anything or they did not yet look at what I thought was most relevant, then I would suggest for them to consider what I thought was relevant. Sometimes the student could use what I suggested and other times they could not. If the student was struggling with using the resource I referenced, I would continue to work through why the example could be useful to see if they could ever connect why I would use that part of their notes. Overall, this provided insight into the problems that a student may not need notes for and what problems a student may want notes for. If the student wanted notes, there was an opportunity to observe what the student deemed possibly relevant to the problem. If they found something they thought was relevant, then there was a chance to see if they could evaluate whether or not it was relevant. Lastly, if provided something relevant according to the interviewer, then it could be observed whether the student could make connections from what was provided to the problem.

Data Analysis

For each student, I made an initial profile based on the experience of the interviews. These profiles were made by considering each problem the student did and how they used resources during each one. Each interview consisted of about 5 problems and most students had 5 interviews providing me with approximately 25 questions to use as my unit of analysis per student. This provided a frequency on the ways they used their resources which provided insight into the tendencies of each student.

Students' resource use was broken down per task and I coded each student attempt at a task using the following categories: (a) Did not use any resources, (b) referenced class notes, (c) made their own example, (d) looked for an example or tried to make their own but failed, and (e) could or could not use an example provided by the interviewer. Of these actions, they also have varying degrees of success as well as different ways of being performed. However, these cover the actions around resources that students could perform. A frequency table was generated showing how frequently each student performed any of those actions. With the frequencies, I sought to identify whether each student had a recurrent style of resource use, addressing Research Question 1. As a note, the students were not asked directly to describe their own resource use style.

After this, I examined students' resource use by tasks. Though not all students were assigned the same problems, the topics were largely consistent among all the students. The topics mostly spanned set theory, like supremum and infimum, and sequences. Thus, the data gathered covers similar problems for each student. This uncovered that some problems tended to have the students perform similar actions to one another even for students that did not typically behave in that way. This suggests that the task itself has influence on how the student will decide to tackle the problem regardless of their usual tendency. The characteristics of the task itself as well as how similar tasks may have been presented in class both serve as factors towards addressing Research Question 2.

Results

The following are a few exemplary student cases that show how students can adopt different styles on their usual resource use when attempting homework tasks.

Jordan

Jordan frequently looked through their notes for help on their homework tasks. On several occasions, Jordan was able to adapt proofs efficiently and effectively from class into solutions for their homework tasks. Here is an example: Given a set $E \subset \mathbb{R}$ and $r > 0$, define the set $rE = \{rx : x \in E\}$. In Jordan's notes, they found where the instructor proved that: $\sup(E + r) = \sup E + r$ for the set $E + r = \{x + r : x \in E\}$. The following table provides the proof that the instructor gave during class and how Jordan adapted it to solve the problem.

Table 2: Instructor vs Student Proof for a Similar Task

Instructor's Work	Jordan's Work
Assume that $a \in E + r$ Then, $a = x + r$ for some $x \in E$. Since $\sup E$ is an upper bound of E, $x \leq \sup E$ so $a = x + r \leq \sup E + r$. Thus, $\sup E + r$ is an upper bound of $E + r$. Now assume N is an upper bound of $E + r$. Let $x \in E$. So $x + r \in E + r$ so $x + r \leq N$. Then $x \leq N - r$. Since $N - r$ is an upper bound of E, $\sup E \leq N - r \Rightarrow \sup E + r \leq N$ Then $\sup E + r$ is the supremum of $E + r$.	Assume $a \in rE$. Then, $a = rx$ for some $x \in E$. Since $\sup E$ is an upper bound of E, $x \leq \sup E$ so $rx \leq rsupE$. Thus, $rsupE$ is an upper bound of rE. Now assume N is an upper bound of rE. Let $x \in E$. So $rx \in rE$ so $rx \leq N$. Then $x \leq \frac{N}{r}$. Since $\frac{N}{r}$ is an upper bound of E, $\sup E \leq \frac{N}{r} \Rightarrow rsupE \leq N$. Then $rsupE$ is the supremum of rE.

When working on this problem, Jordan read their notes and interchanged the parts as they read. Jordan was able to write the proof without assistance from the interviewer. This illustrates how Jordan was able to both recognize a relevant example to adapt as well as an ability to recognize what parts of that proof were interchangeable with different characteristics. I coded this as Resource as Servant. Jordan both identified the relevant proof as well as efficiently identified all the relevant parts that should be changed based on the context. The notes served as an outline, but Jordan had the knowledge on how to fill in the blanks or change the instructor's work.

Peyton

Peyton did not frequently use their notes. However, they did reference them a few times. More frequently, Peyton thought up their own *conceptual examples* or recall *conceptual examples* from class. Conceptual examples refer to specific cases from which Peyton would try to generalize from. Such as using the interval $[2,3]$ to reason why a closed, bounded interval must contain its supremum. They would try to use those to reason through what they needed to do with their homework tasks. Occasionally, this strategy showed a lot of power for certain tasks. However, at other times this technique was not effective enough for Peyton to produce a proof.

One instance of Peyton using their notes was when asked to prove that $x_n = 2n + 6$ is a strictly increasing sequence. Peyton initially looked up where the instructor talked about monotonicity, but they did not find an example they judged to be similar. During the second

phase of this task, I suggested to look at a proposition where the instructor proved that $\{x_n\}$ is non-decreasing if $x_n \leq x_m$ for all $n \leq m$. Peyton noted that the instructor used induction, so they tried to implement induction for this problem. With this approach, Peyton noticed that: $x_n = 2n + 6 \leq 2(n + 1) + 6 = x_{n+1}$.

Peyton finished up their argument by establishing that $P(1) < P(2)$ and, by the logic before, $P(n + 1) < P(n + 2)$. After completing this, Peyton decided that a direct proof just stating that $x_n = 2n + 6 < 2n + 2 + 6 = 2(n + 1) + 6 = x_{n+1}$ was a sufficient proof.

I coded this proof construction as Resource as Partner. Peyton initially could not find any examples they thought would help to solve this problem. With my suggestion, Peyton was able to find an instance where they did something sort of similar to this problem. Peyton noted that the instructor used induction and decides to take that as a suggestion to do that themself. With this suggestion, they were able to make a satisfactory solution. However, afterwards, they noted that they do not think induction is necessary. This suggestion at the end is what distinguishes this resource use as Resource as Partner rather than Resource as Master. Peyton took the suggestion from their notes, but they keep a critical eye on what we're doing.

Charlie

Charlie frequently did not use their notes nor construct their own examples. The frequency analysis of their task solutions revealed that they rarely used their notes and often attempted to reason using the definitions and theorems presented in class. Occasionally, Charlie would construct their own examples, but this usually did not yield a valid solution to the task. When they ran into obstacles, I would provide Charlie with examples from the class or create a unique example to reason with, but they frequently did not use these to produce a valid task solution. Charlie usually mentioned that they wanted to go talk to the instructor during office hours to get help when they got stuck. This overall disinterest in using their notes or reasoning with examples contrasts with all the other students. However, in many cases, Charlie was successful at making strong reasoning connections between the definitions and theorems from class and the problem. With these connections, Charlie demonstrated throughout the interviews that they could construct proofs in real analysis without using their resources for many problems. However, there were several times that Charlie got stuck, but rather than using their available resources at the time instead opted to visit the instructor. For the purposes of this investigation of example use, I found that this student showed individual preference against using examples. However, there were a few occasions where they did opt to use examples, which I will discuss next.

There were only two instances where Charlie directly used notes. One of these instances was for the first tasks asking to prove a limit of a sequence by definition. The other time was to answer the following question: Suppose $S \subset \mathbb{R}$ is nonempty and bounded below and let $v = inf\ (S)$. Prove the following statement: For every $y > v$, there exists $s \in S$ such that $s < y$.

Charlie noted that the instructor proved something similar for the supremum. They changed necessary components to achieve a correct argument. This time the instructor's proof was the analogous supremum problem. As such, the student had to change very little, but Charlie did both identify the relevant proof and adapted it effectively for this context. Similar to the first example, I coded this as Resource as Servant.

Summary of Student Cases

For the students in this study, there were no particular moments where the students seemed to demonstrate Resource as Extension of Self. As such, there is no example given here. Resource as Extension of Self would look like a student bringing up a specific instance where a process was

showcased by the instructor, but the student did not necessarily need it to complete the problem. Rather, the student would like to have the resource available in case they need just a reminder. In some sense, the resource would act as an extension of memory rather than a tool needed to help actively.

Additionally, it is important to end off by noting that all the students could change their position to their resources from problem to problem. For instance, every student faced with their first sequential limit problem to solve using the definition of convergence used their Resources as Master. Every student flipped to an instance where the instructor proved the limit of a sequence by definition and used that as a line-to-line inspiration of their approach. I conjecture that this is due to how the instructors emphasized the routine of this type of proof. When doing these proofs in class, the instructors tackled the problems using the same routine and would always mention that this is how the definitions proof work. While this can sound similar to the Resource as Servant examples from above, these students approached the problem differently in these cases. They acknowledged that they were relying on the instructor's notes to complete the problem. Before, they knew those problems held everything necessary to prove the problem, so they strategically used that structure to make sure their proof was complete. Here, they did not know how to write the proof, so they needed to rely on everything from the notes to complete their proof. This shows that students can renegotiate their position with their resources as they realize different limitations in their preparedness for certain tasks as well.

Discussion

For this study, the class notes were the standard that the student resource use was considered against. The class notes could be divided into providing Conceptual Examples and Proof Examples. Conceptual being the specific examples such as $(1,3)$ as an open interval, and Proof Examples being any instance of a complete proof provided in class. When considering these types of resources, the uses of them can be restrictive. From the observations made, the four categories that the framework considers are sufficient to categorize the resource uses. Largely, Proof Examples are the main resource used by students for their homework. As such, for proof-based mathematics, the resource use framework here could be a useful tool to describe how students are interacting with their available resource in a certain context since it was adapted to fit this situation.

The analysis brings forth a few details about how students use their resources while working on homework tasks. Namely, real analysis students adopt typical styles on how frequently they use their direct class note resource. Some students use their notes at a much higher frequency than others. This frequency might be tied to how confident a student is in their capability to adapt what they read in their notes. This can be partially substantiated by the fact that when I would bring up aspects from class, students that would frequently not reference their notes tended to have a harder time adapting the resources brought forth to them. For instance, Charlie would not commonly use the resource I mentioned to them. Instead, Charlie would frequently continue thinking about the problem using just the definitions and theorems they were already considering. On the other hand, Jordan frequently used their notes to complete their homework and could commonly take at least one useful part from any resource that I offered to them.

A second result of this data is that some types of questions tend to influence students to seek out help from their resources regardless of their previously established style. For instance, when students were asked to prove a limit of a sequence using the ϵ-k definition, every single student looked up an example from class and attempted to copy that structure for the homework task's sequence. I conjecture that this results from how the instructor routinized the problem

during the instruction, or the very specific technique the instructor required the problem to be completed using. With this, the students might have decided that their best strategy would be to attempt an adaptation of what the instructor was doing in-class. In turn, this leads them to return to their resources and attempt to use them as closely as possible. This could influence them approaching the Resource as Master.

Conclusion

In conclusion, students can use their resources in various ways. The way that students use their resources can even vary from problem to problem and context to context. This suggests that students may develop an overall style to how they approach their work, but the way that questions are worded or what content they are covering may change how the student decides to approach the problem than they usually would. One of the strong ways that an instructor may be able to influence resource use of the students is by emphasizing a routinization of certain problems like sequential limits. This can be beneficial when the instructor wants students to solve problems in certain ways. By both verbally reinforcing the routine being used as well as repeated demonstration of the routine in use can help bolster a connection for the students between the routine and the types of problems it is relevant for in a student's problem situation image. For students, knowing how to adapt a proof is a key competence for being successful in a class like real analysis. This is not something emphasized in the literature, but it may be an important part of learning at this stage in a student's apprenticeship in proving.

Future research into the link between routinization and resource use by students could provide interesting results into how instruction could be meaningfully designed. Also, this could provide more insight into how problems that share structure may not be seen as routinized by the students.

References

Anastasakis, M., Robinson, C. L., & Lerman, S. (2017). Links between students' goals and their choice of educational resources in undergraduate mathematics. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *36*(2), 67-80.

Galbraith, P., Goos, M., Renshaw, P., & Geiger, V. (2001). Integrating technology in mathematics learning: What some students say. In J. Bobis, B. Perry, & M. Michael Mitchelmore (Eds.), *Numeracy and Beyond. Proceedings of the 24th Annual Conference of the Mathematics Education Research Group of Australasia*, 225-232.

Gueudet, G., & Pepin, B. (2018). Didactic contract at the beginning of university: A focus on resources and their use. *International Journal of Research in Undergraduate Mathematics Education*, *4*, 56-73.

Muir, T. (2014). Google, Mathletics and Khan Academy: students' self-initiated use of online mathematical resources. *Mathematics Education Research Journal*, *26*, 833-852.

Ní Shé, C., Mac an Bhaird, C., Ní Fhloinn, E., & O'Shea, A. (2017). Students' and lecturers' views on mathematics resources. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *36*(4), 183-199.

Pepin, B., & Kock, Z. J. (2021). Students' use of resources in a challenge-based learning context involving mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *7*(2), 306-327.

Radović, S., & Passey, D. (2016). Digital resource developments for mathematics education involving homework across formal, non-formal and informal settings. *The Curriculum Journal*, *27*(4), 538-559.

Selden, A., Selden, J., Hauk, S., & Mason, A. (1999). *Do Calculus Students Eventually Learn to Solve Non-Routine Problems?* Department of Mathematics, Tennessee Technological University.