Finding Elements of Exploration in a Ritualized Calculus Discourse

Mark Watford
Florida State University

Undergraduate mathematics classrooms continue to experiment with active learning strategies as an alternative to the traditional lecture-based teaching model. This paper investigates one student's (Jacob) engagement in a poster session activity in a Calculus 1 course and explores how he reasoned about calculus concepts in this non-standard learning environment. Informed by Sfard's theory of commognition, an analysis of Jacob's discursive routines reveals a complex interplay between ritualized and explorative discourse. While Jacob's poster presentations appear highly ritualized on the surface, with routines recycled from class, indications of explorative activities emerged in his preparation with a partner, reflecting an active search for suitable routines. This research emphasizes the importance of considering the entire ritual-exploration continuum in mathematics education and raises questions about facilitating the transition from ritualized to explorative discourse.

Keywords: Commognition, Routines, Calculus, Exploration, Rituals

Introduction

In undergraduate mathematics education, there is a growing trend of incorporating forms of learning which differ from the standard lecture model. One notable study reported calculus instructors being encouraged to employ *active learning* strategies in their classroom (Rasmussen et al., 2014) as one of seven characteristics of successful calculus programs of doctoral granting institutions in the country. However, the term *active learning* is vague and has varying interpretations. Generally, it can refer to any form of student engagement in class which differs from the standard lecture model, ranging from the incorporation of worksheets to engaging in group discussions and problem solving (Freeman et al., 2014). Nevertheless, there have been recent calls to incorporate such strategies in tertiary mathematics learning (e.g., CBMS, 2016; Saxe & Braddy, 2015) and a call to investigate such "interventions in practice" (Rasmussen et al., 2014, p. 509). At Barrow University[1], a large research-intensive university in the southeastern United States, one intervention has been implemented in the Calculus 1 courses for several years and serves as the context for this research project. The weekly activity of students engaging in poster sessions (further described in the "Methods" section) was designed to have students become expert at solving one problem and then communicating their solution to their peers. Because communication is at the forefront of this activity, Sfard's (2008) theory of commognition, which considers thinking is communicating, is fitting to study how students reason about calculus concepts, especially in a non-standard learning environment. For the purposes of this paper, I focus on the discursive routines of one student, Jacob, and how he situated the poster session activity as explorative or ritualized. Preliminary findings indicate that the presentation of the poster was highly ritualized, yet there were moments of exploration evidenced in Jacob's discourse when speaking with his instructor. As well, there was also an indication that Jacob engaged in more explorative routines when designing the poster he presented, but the constraints of the assignment may have funneled Jacob into a ritualized routine.

[1] All names in this paper are pseudonyms.

Literature Review

Commognition has gained popularity recently with the framing's tenet that students' thinking is revealed in their communication. Sfard (2016) promoted that analyzing discursive activities stands in stark contrast to a more cognitivist approach which has "the tendency to view communication as a mere window to something else" (p. 41). With a commognitive lens, thinking *is* communicating and vice versa. Since its inception, a wide range of aspects of commognition have been trialed in various forms in the research of undergraduate mathematics education, particularly with intervening learning activities that disrupt the standard lecture model. Most recently, Kontorovich (2023) critically examined the panacea-like collaborative learning structure by attending to a triad of cognitive, social, and affective aspects of students' collaboration as they studied linear algebra. His research explored how two students, Jan and Sai, were able to position themselves as mathematizers (or not as mathematizers) and how they perceived each other as such. The careful commognitive analysis of the dynamics between the two students illuminated a great need for further research into how the triad of mechanisms plays a role in collaborative learning situations.

Another avenue of study in which the lecture model is disrupted through varying forms of classroom activity is the continuum of routines. Lavie et al. (2019) operationalized routines with a commognitive lens and made a distinction between rituals (process-oriented routines) and explorations (product-oriented routines). Whereas rituals typically are guided by the question, "how do I proceed," explorations are guided by the question, "what is it that I want to get" (Lavie et al., 2019, p. 166). Rituals should be in no way absent from a novice mathematics learner's discourse. They are the foundation of transitioning to an explorative discourse. Therefore, as a mathematics learner becomes a full-fledged member of the mathematics community, they undergo a gradual de-ritualization of their routines and move toward the exploration end of the spectrum. In fact, certain curriculum materials which are "explicitly designed to promote student-centered activities that shift classroom discourse away from lectures" (Barnett, 2022, p. 1579), such as Primary Source Projects (PSPs), were analyzed for their potential to de-ritualize mathematical discourse at the tertiary level. Barnett (2022) showed how the task situations in PSPs ranged on the whole of the ritual-exploration continuum and may be used to fuel the de-ritualization process (depending on how the PSPs are implemented).

Similarly, Nachlieli and Tabach (2019) and Sfard (2016) proposed that the nature of routines in which students engage is highly dependent on the ways in which a lesson or activity is implemented. Nachlieli and Tabach (2019) developed a framework to analyze mathematical learning tasks as either ritual-enabling opportunities to learn (OTL) or exploration-requiring OTL. They, too, stated that rituals are essential for students to enter the discourse because rituals help to develop students' object-level and meta-level discourse which is the foundation of explorative activities. However, Sfard (2016) analyzed the discursive routines of a teacher during a lesson and subsequently the routines of his students as they solved similar problems. She found that they mirrored his ritualized routines and exhibited some of the same features of his routines. Thus, it is not surprising that it should be essential to consider the routines afforded by and exhibited in such classroom activities.

Theoretical Framework: Commognitive Routines

Researchers of mathematical discourse are no stranger to Sfard's (2008) theory of *commognition*, a concatenation of *communication* and *cognition*, which posits that thinking and communication are one in the same. Sfard (2008) proposed that mathematical discourse is a special form of discourse that can be characterized by four properties: word use, visual mediators, narratives which can be composed of key words and visual mediators, and routines. Both word use and visual mediators may be rather trivially conceptualized as those special words and symbols we use to communicate mathematics and which are exclusive to mathematical discourse. For instance, in the context of evaluating a limit, key words and visual mediators may include the words *indeterminate form* and algebraic symbols like $\frac{\infty}{\infty}$ or graphs that show the end behavior of a function. Narratives are the stories we tell about mathematical objects using the key words and visual mediators (e.g., $\lim_{x \to \infty} \left(\frac{2^3}{3}\right)^x = \infty$ because $\frac{2^3}{3} > 1$). Finally, routines are the repetitive patterns found in mathematical discourse (e.g., evaluating a limit using algebra). For the purposes of this paper, routines will be characterized as explorative or ritualized.

Explorations may be considered by many in the professional mathematical community as the true form of "doing mathematics." That is, the goal of an exploration is to produce a narrative about mathematical objects which the mathematical community generally accepts as true. Thus, explorations may be considered as product-oriented routines (Lavie et al., 2019). On the opposite end of the routinization spectrum is the ritual. Those who perform rituals are completing discursive actions with the goal of "creating and sustaining a bond with other people" (Sfard, 2008, p. 241) and may not be concerned about telling stories about mathematical objects. A ritual is "appreciated for its performance and not for its product" (Lavie et al., 2019, p. 166) and therefore is considered a process-oriented routine.

Because pure rituals and explorations are truly rare in mathematics classrooms (Lavie et al., 2019), I will characterize elements of the mathematical discourse of a student presenting a poster to his peers and instructor as ritualized or explorative and seek to answer the question:

How might a student engage in discursive routines in the context of a Calculus 1 poster session?

Methods

Context of the course and poster session

Data from this study were collected from a Calculus and Analytic Geometry 1 course taught in the Spring 2023 semester at a large southeastern research-intensive university. The mathematics department employs graduate teaching assistants as instructors of record for the four-credit hour calculus course as well as a specialized teaching faculty member who serves as the course mentor and collaborates with TAs to implement a unified curriculum. Members of the teaching team are expected to adhere to the same schedule, covering specified content and implementing a hallmark classroom activity, poster presentations, weekly. The poster activity consists of students collaborating with a partner outside of class to solve one calculus problem and then presenting their solution as a poster during class. The problems on which students collaborate are comparable to what they may be given as a homework problem and are directly related to the content of a recent lecture. As stated in the activity description given to students, they are expected to become "expert" at solving the problem. Their expertise is demonstrated through the inclusion of four mandatory components of the poster: detailed steps to solve the problem, the "final answer" to the problem, clues from the problem that may lead to the solution, and mistakes made or mistakes students anticipate others might make while solving the problem. Once per week, the class engaged in two rounds of poster sessions. In the first round, one person from each pair stood at their respective poster while the partner visited other posters. The person standing explained their problem to anyone who visited and should have been able to answer any questions posed by their visitors. After 10–15 minutes, partners switched places, and the second round commenced. The poster session from which this analysis was taken occurred in the third week of the semester and consisted of problems related to calculating limits and finding the derivative at a point.

Data Collection

The episode analyzed for this paper is part of a larger research project in which 24 of 29 students consented to have their classroom interactions audio/video recorded and copies of their coursework collected as data. Field notes and recordings of the class lectures were captured. Additionally, six of the 24 consenting students participated in an interview designed to expand on their mathematical routines observed in the class. This paper focuses on one consenting student, Jacob, who also participated in an interview. Although Jacob initially enrolled in the course to be eligible for entrance in the computer science program, he revealed that after a few weeks of being in the course, he realized that he wanted to progress in a different direction entirely and major in Chinese Language and Culture. By personal admission, Jacob said that a heavy course load was one factor contributing to his change of heart. The other factor was a newfound interest in learning about Chinese culture. The episode of analysis is from Jacob's second poster session in which he was tasked with evaluating the limit using algebra: $\lim\limits_{x\to\infty} \frac{2^{3x+1}}{3^{x+3}}$.

Data Analysis

When analyzing the poster episode, I drew upon Sfard's (2016) and Lavie et al.'s (2019) notion of the routine as a spectrum ranging from ritual to exploration. According to Sfard (2016), there are three main features of a routine that need to be considered when determining whether an episode may be explorative. First, the subjects of the narratives are typically abstract objects rather than more concrete symbols and signifiers. Second, the source of mathematical narratives in an explorative discourse "can be logically deduced from stories already endorsed" (Sfard, 2016, p. 45) as opposed to relying on memory or an authoritative source, such as the instructor or textbook. Lastly, the goals of mathematical activity must be considered. As described above, the goal of an exploration is to tell a story about mathematical objects while the goal of a ritual is to engage in the process itself. The data sources which inform the following analysis are the interview transcript, the poster presentation transcript, and the poster artifact.

Results

The following episode is taken from Jacob's poster presentation in which he presented to 13 persons including the instructor (Ms. Rebecca) and the researcher (Mark). Jacob's poster monologue was roughly the same for each presentation. He admitted in the interview that he did not rehearse a script, however, he said:

> as the poster session gets going, I do say like a similar spiel…. obviously, the prompts are laid out step by step; I go step by step. And, I say pretty much the same thing, although sometimes I change the wording a bit to make it a bit more interesting to myself.

Therefore, the components of his mathematical discourse are virtually identical in each presentation. Below is one representative example of Jacob's presentation which he would begin once a visitor appeared before him.

We have [the problem][2] find the limit of x to infinity of two to that over three to that [pointing in the general area of the upper left-hand corner of the poster]. So yes, and put very simply, no matter what we tried, we ended up with the indefinite[3] form. So, we went [with] *undefined*. We tried to divide by the conjugate because of those pesky exponents. And yeah, we still got infinity over infinity, so we went with *undefined*. To be honest, [we are] not sure if it's correct.

Figure 1 displays the poster that Jacob and his partner, Rachel, created to present their solution to the limit problem.

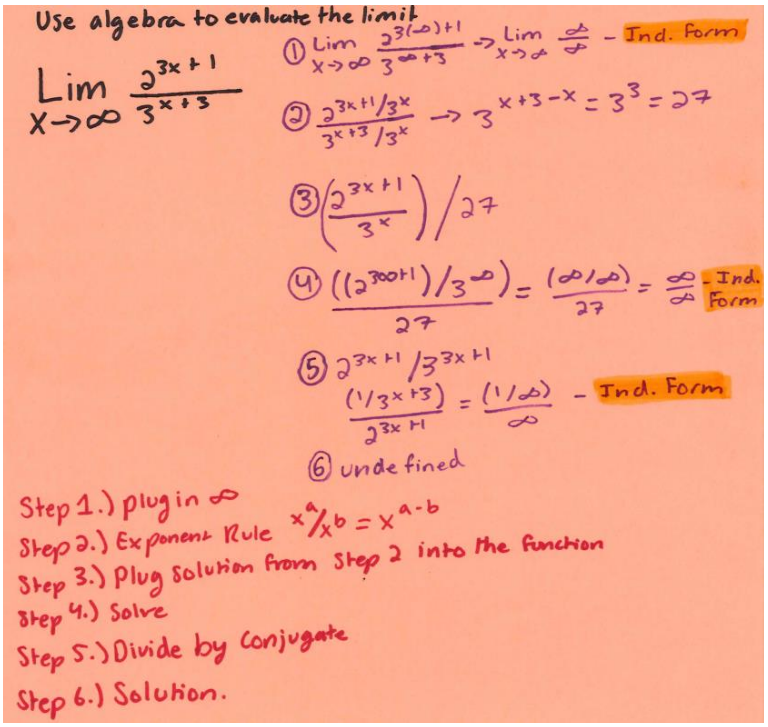

Figure 1. Jacob and Rachel's poster detailing their three attempts at evaluating the limit.

Objects of Mathematical Discourse

One distinctive component of mathematical discourse is characterized by how discursants communicate about mathematical objects. The objects of mathematical discourse were coded as either concrete or abstract. This distinction was dependent not only on the key word itself but also how the discursant communicated about the object. In other words, concrete objects were treated by the discursant as something tangible and that could be operated on by a person, whereas abstract objects existed primarily in the discourse and could take on discursive actions. For example, Jacob spoke about $\lim_{x\to\infty} \frac{2^{3x+1}}{3^{x+3}}$ as something "we have" in terms of a concrete problem which needed a solution. There is no perceptible indication that Jacob considered the

[2] Words in brackets clarify Jacob's meaning based on his other instances of delivering virtually the same monologue.

[3] Jacob consistently used "indefinite" in place of "indeterminate." This seemed to have no bearing on what Jacob meant by the "indefinite" form. In fact, his instructor and his partner both used this term when speaking about the indeterminate form while reasoning through Jacob's poster. However, the instructor did not use "indefinite form" while lecturing.

function inside the limit notation as a quotient of exponential functions with certain properties. In fact, Jacob further concretized the limit in his recitation of the function by referring to the exponents as “that” and “that.” The reason for this word choice is not apparent; however, I conjecture that this was a form of colloquial discourse in which Jacob felt comfortable engaging with his classmates. This is supported by the fact that Jacob changed his discourse once I (who established myself as a researcher in the class) visited his poster. While explaining the poster to the me, Jacob recited the entirety of the exponents as “two to the three x plus one over three to the x plus three.” Jacob may have perceived me as a person with whom it would be inappropriate to engage in colloquial discourse. However, once I left, Jacob adopted this practice of reciting the exponents as he did for me in subsequent presentations to both students and the instructor. Regardless of the reason for the word choice, Jacob still treated the exponents as “things” rather than mathematical objects with special properties.

Perhaps the most concrete of objects Jacob mentioned were “those pesky exponents.” Jacob spoke of the exponents in terms of being physical obstacles imbued with the ability to cause trouble. More characteristic of an explorative discourse would be how Ms. Rebecca spoke about the exponents when she was trying to help Jacob reason through evaluating the limit. Ms. Rebecca posed the question, “are the exponents actually growing faster in the top or the bottom [of the fraction]” as if to model the types of questions her students should be asking when solving a problem. The way in which she spoke about the behavior of exponents and how that behavior affects the overall function is more indicative of discourse with abstract objects. Although Jacob’s discourse appeared highly ritualized in the poster sessions based on how he communicated about mathematical objects, the source of his narratives and the goals of his mathematical activity may shed light in a different direction.

Sources of Mathematical Narratives

The source of Jacob’s mathematical narratives both within the poster and the poster session itself was predominantly a recycled routine. This is more indicative of ritualized routines. Jacob’s poster showed that he and his partner tried to recycle three routines they learned in class: substituting a value for the variable, using exponent rules to simplify the expression, and multiplying by a quotient of conjugates. Although it may be a mere observation that Jacob and his partner attempted to recycle three routines, this was confirmed in Jacob’s admission to his instructor that he was looking for the appropriate routine to recycle but did not find one that matched the current situation. He said, “I don’t know if we’ve had a problem where we just straight up said *undefined*; so, that’s why I was kind of suspicious.” This explains why Jacob felt the need to end each of his presentations with a word of caution about the accuracy of his solution.

Moreover, Jacob verified this in his interview as an almost instinctive normal practice when completing the poster each week. When working on a poster problem, he would sometimes refer to his notes, “comparing it to previous examples of problems” which were similar or “problems that use similar methods.” This may not be an unfamiliar practice to professional mathematicians nor to the commognition literature (see Lavie et al.’s (2019) discussion on *precedents*), but Jacob’s seemingly automatic trial of the three routines based on key features of the problem (the presence of infinity, exponents, and a sum in a quotient function) suggests that he was acting without much purpose. Furthermore, it appears from “Step 5” (Figure 1) that Jacob and his partner were not entirely certain of what a conjugate is or how it may be used to help evaluate limits. This may be typical of a novice in the mathematical discourse who recognizes that some discursive action is necessary, but it is not apparent what the discursive action should be.

In contrast to Jacob's seemingly highly ritualized poster presentation, there was some indication that he engaged in a more explorative activity when preparing the poster with his partner. Jacob consistently said that he "tried" different routines showing that the source of these narratives was not entirely automatic recall; rather, he and his partner were actively searching for an appropriate routine that corresponded to a non-indeterminate form solution. Because the routine of ending with an indeterminate form was unacceptable according to Jacob's previously encountered routines, he knew that he needed to engage in a different routine. The actual "how" of that routine and "which" routine was unclear to him. This trialing of routines shows a mixture of Jacob asking the questions that Lavie et al. (2019) characterized as explorative and ritualized. Jacob appeared to be asking *how to proceed to get* a non-indeterminate form.

Goals of Mathematical Activity

The third aspect of a routine that can serve to distinguish between explorative and ritualized discourse is the goal of mathematical activity. According to Sfard (2016), the objective of performing a ritual is grounded in establishing or developing a relationship with another person and adhering to the goals set by others. The objective of engaging in an exploration is to know more about mathematical objects" (p. 44) and produce an endorsable narrative about those objects. The explorative goal is more intrinsically motivated while extrinsic motivation drives rituals. It is possible that Jacob and his partner "went [with] undefined," because providing a solution was one of the goals of the assignment. This is apparent because in four of Jacob's presentations, he ended with "we went with what we thought was best." In contrast, Jacob's discourse with his instructor showed that he genuinely wanted to know more about evaluating the limit in this case. Jacob began his poster presentation to his instructor by saying, "I'm pretty sure this is incorrect, so that's why I want to present it to you: to see if you can explain." There was no precedent that Jacob would be able to modify his poster to improve his grade, so the desire to know how to evaluate the limit in this case was more of an intrinsic motivation to simply "know more." Further evidence of this conjecture was found in Jacob's interview. When asked why he attempted to solve the poster problem, one of the reasons that Jacob gave was "part of it is showing to myself that, hey, I can actually do this stuff. You know, I'm actually not that bad at math, even though I feel like I am." The goal of mathematizing to establish and develop a relationship with his mathematical self is, by definition, ritualized discourse. Yet, it is apparent that the explorative spark is rooted in Jacob's discourse, and the ritualized discourse in which he engaged during the poster session was not wholly absent of mathematical exploration.

Discussion

This research offers insight into the complex nature of student engagement with mathematical concepts at the tertiary level and the role of discourse in shaping students' mathematical thinking. It underscores the importance of considering the entire ritual-exploration continuum in the mathematics classroom and highlights the interplay between the two extremes. Future research in this area could explore how instructors can design activities that intentionally promote the transition from ritualized to explorative discourse, such as PSPs (Barnett, 2022) or other exploration-requiring OTL (Nachlieli & Tabach, 2019). Additionally, investigating how students' discursive routines evolve over time as they progress in their mathematics learning would provide a deeper understanding of the developmental aspects of mathematical thinking.

References

Barnett, J. H. (2022). Primary source projects as textbook replacements: A commognitive analysis. *ZDM – Mathematics Education*, *54*(7), 1569–1582. https://doi.org/10.1007/s11858-022-01401-2

Bressoud, D., & Rasmussen, C. (2015). Seven characteristics of successful calculus programs. *Notices of the American Mathematical Society*, *62*(2), 144–146.

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 685–698. https://doi.org/10.1080/0020739X.2013.798874

The Conference Board of the Mathematical Sciences (CBMS). (2016). *Active learning in post-secondary mathematics education*. https://www.cbmsweb.org/2016/07/active-learning-in-post-secondary-mathematics-education/

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences of the United States of America*, *11*(23), 8410–8415. https://doi.org/10.1073/pnas.1319030111

Kontorovich, I. (2023). When learning stumbles upon identity and affect: A loaded student-student collaboration in linear algebra. *International Journal of Mathematical Education in Science and Technology*, *54*(8), 1526–1540. doi.org/10.1080/0020739X.2023.2173102

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153–176. https://doi.org/10.1007/s10649-018-9817-4

Nachlieli, T., & Tabach, M. (2019). Ritual-enabling opportunities-to-learn in mathematics classrooms. *Educational Studies in Mathematics*, *101*(2), 253–271. https://doi.org/10.1007/s10649-018-9848-x

Rasmussen, C., Apkarian, N., Hagman, J. E., Johnson, E., Larsen, S., Bressoud, D., & The Progress through Calculus Team. (2019). Characteristics of precalculus through calculus 2 programs: Insights from a national census survey. *Journal for Research in Mathematics Education*, *50*(1), 98–111. https://doi.org/10.5951/jresematheduc.50.1.0098

Rasmussen, C., Marrongelle, K., & Borba, M. C. (2014). Research on calculus: What do we know and where do we need to go? *ZDM – The International Journal on Mathematics Education*, *46*(4), 507–515. https://doi.org/10.1007/s11858-014-0615-x

Saxe, K., & Braddy, L. (2015). *A common vision for undergraduate mathematical sciences programs in 2025*. The Mathematical Association of America. http://www.maa.org/sites/default/files/pdf/CommonVisionFinal.pdf

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sfard, A. (2016). Ritual for ritual, exploration for exploration: Or, what learners are offered is what you get from them in return. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 41–63). Routledge. https://doi.org/10.4324/9781315643236