The Evolution of Two Undergraduates' Example and Set Use During Conjecturing and Proving

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To better understand how students' example and set use might evolve during their conjecturing and proving activity, we engaged two students in guided reinvention of mathematical statements relating sequence properties during an 11-week teaching experiment. We characterized the students' evolving example/set use in three categories: (1) classifying examples from the initial set of sequences, (2) seeking diversity and using lack of examples from an expanded initial set of sequences, and (3) attending to properties, searching for structure, and building formality with the set of all sequences. We exemplify these categories and then discuss some guidance that could explain the students' example/set use.

Keywords: Examples, sets, conjecturing, proving, sequences

Studies show that students find conviction and justify conjectures by using examples (Coe & Ruthven, 1994; Healy & Hoyles, 2000) and some suggest that students need to be supported to recognize limitations of examples (Knuth, 2002; Martin & Harel, 1989; Stylianides & Stylianides, 2009). More recent research examines how example-based reasoning can support deductive reasoning (Aricha-Metzer & Zaslavsky, 2019; Iannone et al., 2011; Komatsu et al., 2017; Ozgur et al., 2019; Pedemonte & Buchbinder, 2011; Sandefur et al., 2013), including how mathematicians and students use examples in their proving activity (Alcock & Inglis, 2008; Ellis et al., 2019; Inglis et al., 2007; Lockwood et al., 2016; Lynch & Lockwood, 2019; Weber, 2008). Furthermore, researchers have drawn on cognitive unity to explore how the development of students' conjectures based on their work with examples is reflected in the students' respective proving activities and proofs (Lin & Wu, 2007; Pedemonte & Buchbinder, 2011). Additionally, Fiallo and Gutiérrez (2017) studied what types of conjectures and corresponding proofs students developed after experimenting with Dynamic Geometry Software. This research points to the potential usefulness of examples for students in their conjecturing and proving activities.

Additionally, a growing amount of literature shows the usefulness of sets for students as they make sense of mathematical statements and engage in proving activities (Dawkins, 2017; Dawkins & Cook, 2017; Hub & Dawkins, 2018). For instance, Dawkins (2017) explained that it can be productive to consider a subset relation when making sense of a mathematical statement in the form *if p, then q*; that is, the set of all objects that fit property p is a subset of all the objects that fit property q. However, little research has explored how students can use sets to author and prove their conjectures.

To better understand how students' example and set use might evolve during their conjecturing and proving activity, we engaged two students in guided reinvention (Freudenthal, 2005; Gravemeijer, 1999; Gravemeijer & Doorman, 1999) of mathematical statements relating sequence properties during an 11-week teaching experiment (Steffe & Thompson, 2000). In this study, we examined the evolution of two calculus students' example/set use in response to an intervention we crafted for supporting students' conjecturing and proving activity. We investigated: How can students use examples and sets to support conjecturing and proving? Specifically, how did a pair of students use sets and examples as they were guided to reinvent mathematical statements relating sequence properties?

Theoretical Perspective

We investigated how two calculus students' example and set use evolved as they engaged in conjecturing and proving. By conjecturing, we mean developing a statement that relates mathematical concepts the student suspects are true but does not yet know are true (Lannin et al., 2011). In this study, we are particularly interested in students' conjectures from observing consistent patterns in a finite number of cases (Cañadas et al., 2007) and the generality of the students' conjectures (that is, whether the conjectures were specific to an initial set of finite cases or made a claim about additional cases). The cases that the students in our study observed were instances of sequences (e.g., $x_n = n^2$). Additionally, we use proving to mean conveying a general argument, from the students' perspective, for the statement's truth.

Throughout our data, students organized (and re-organized) examples of sequences into sets. We attended to different sets and set operations to better understand the students' set use. The students' universal set was key to understanding the generality of students' conjectures. By *universal set* we mean the set of all elements under consideration. The students' universal set began as an initial (finite) set of sequences, later expanded to include additional examples, and eventually included all sequences, where the specific examples were just representative members of this infinite set. Additionally, we determined whether students seemed to attend to a union, intersection, subset, complement, and difference (as typically defined in mathematics). For the sake of a relevant example, consider a universal set as the set of all sequences (represented by the shaded region in Figure 1A), which includes all the sequences that are bounded above (purple set in Figure 1), bounded below (green set in Figure 1), or unbounded. The shaded region in Figure 1B represents the *union* of the set of bounded above sequences and the set of bounded below sequences, whereas the shaded region in figure 1C depicts the *intersection* of these two sets. The shaded region in Figure 1D represents the set of sequences bounded below, which is a *subset* of the set of all sequences. The shaded region in Figure 1E represents the *complement* of the set of sequences that are bounded below. The *difference* between the set of bounded above sequences and the set of bounded below sequences is represented by the shaded region in Figure 1F. Finally, Figure 1G highlights that the set of convergent sequences (yellow set) and the set of sequences that are bounded above and not bounded below are *mutually exclusive*. We continue to denote sequences that are bounded above, bounded below, and convergent in this way (i.e., with green, purple, and yellow outlined sets, respectively) throughout our results.

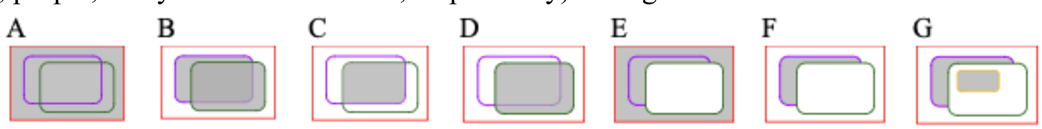


Figure 1. Representation of sets and set operations.

To further understand students' example use, we drew on Ellis et al.'s (2019) Criteria, Affordances, Purposes, and Strategies (CAPS) framework, which offers a way to describe students' example-based activity while investigating and proving conjectures. Specifically, we attended to Purposes and Strategies to understand how students' example-based reasoning supported their conjecturing and proving activity. *Purposes* are the reasons students use examples. For instance, a student might use the example(s) to: form a connection between two concepts (*conjecture development*), confirm their belief in a conjecture's truth (*belief confirmation*), determine a conjecture's truth (*test the truth*), reject a conjecture as true (*refute*), illustrate their claim that a conjecture is true or false (*convey claim*), identify the causal mechanism behind a conjecture (*understand why*), convey an argument supporting their claim that generalizes across cases (*convey a general argument*), and/or respond to a request or prompt (*respond to teacher-researcher*[1]). Strategies are "the range of deliberately strategic approaches students employ both in choosing and using examples" (p. 265). A student can deliberately create a new example that features different properties than previous examples (*seek diversity*), select examples based on particular properties of interest (*attend to properties*), or repurpose an existing example by varying one or more elements (*create systematic variation*). A student can deliberately use examples by searching for a pattern or a mathematical structure to identify general features (*searching for structure*) or developing a formal representation to express what is the same across all examples (*building formality*).

Methods

Data Collection

We conducted a teaching experiment (Steffe & Thompson, 2000) with two students, Lara and Stella (pseudonyms), recruited from a Calculus 1 course at a large university the semester before the experiment. We aimed to experience the students' mathematical learning of and reasoning about sequences and series as they formed conjectures about sequence and series properties, defined related terms, and proved their conjectures. Both students indicated they were freshmen at the time of the experiment, and both used she/her pronouns. Lara was majoring in Biological Chemistry, and Stella was double-majoring in Psychology and Neurosciences. Lara earned a 2.5 (on a 4.0 scale) in her Calculus 1 course, and Stella earned a 4.0. The experiment consisted of eleven 1.5-hour sessions in which the students' collaborative digital work was audio and video recorded. In these sessions, the first author was the teacher-researcher and the third author was the observer.

Realistic Mathematics Education, especially the heuristic of guided reinvention (Freudenthal, 2005; Gravemeijer, 1999; Gravemeijer & Doorman, 1999), framed the instruction during the teaching experiment (similar to Lockwood & Purdy, 2019 and Swinyard & Larsen, 2012). We describe the general task design of the teaching experiment since the details are beyond the scope of this paper. The overarching design goal of the experiment was to guide students to conjecture and prove mathematical statements that relate sequence properties (e.g., all convergent sequences are bounded) and statements about series convergence (e.g., the comparison test). To do so, we engaged the students in context problems (Gravemeijer & Doorman, 1999) with sequences (sessions 1-7) and then with series (sessions 8-11). For this report, we analyzed data from Sessions 4-7, when we asked the students to sort sequences based on different properties (increasing, bounded above, bound below, and converging), make conjectures about sequence properties, and justify these conjectures. The first three sessions mainly focused on the students generating and defining sequences. We refer to these sequences as the students' initial set.

Data Analysis

We transcribed all teaching episodes and enhanced the transcripts that we analyzed for this study with embedded pictures of the students' work. We first identified instances of the students discussing connections between bounded, increasing, and/or convergence by reading the transcript and watching the corresponding video. Once we identified these instances, at least two of the three authors participated in coding the data, with any disagreements resolved through

[1] Ellis et al. (2019) refer to this as "placate the interviewer." We used this different phrasing since it better fit our teaching experiment setting.

discussion. In particular, we investigated the students' example use by identifying purposes and strategies using Ellis et al.'s (2019) framework (see previous section). We also identified additional strategies that we did not think were captured by the framework: *classifying* and *using lack of examples*. We coded data as *classifying* when it featured the student checking whether a sequence fit their concept definition and/or concept image (Tall & Vinner, 1981), which often occurred when students considered moving or adding sequences inside (or outside) a category. We coded data as *using lack of examples* when the students reflected on their inability to create an example with specific properties given the properties' parameters. During this pass of the data, we also considered how the students used sets, looking for evidence of the students considering an *intersection*, *subset*, *complement*, *difference*, and/or if sets were *mutually exclusive* (see previous section). Note that the students did not necessarily (if ever) use words like the "union;" instead, these were our way of denoting the sets that the students seemed to consider. We also searched for evidence of what the students considered their universal set (i.e., the students' initial set of sequences, some other finite set of sequences that expanded their initial set, or the set of all sequences). After we coded each episode, we wrote a vignette of the students' example/set use and compiled the vignettes into one document, organizing them chronologically. We then re-read the vignettes, noting changes in the students' example/set use (asking ourselves how the students' example/set use differed from/similar to the previous episode). Through this constant comparison, we noticed three categories that characterize the students' evolving example/set use, which we exemplify next.

Results

We characterized the students' evolving example/set use in three categories: (1) classifying examples from the initial set of sequences, (2) seeking diversity and using lack of examples from an expanded initial set of sequences, and (3) attending to properties, searching for structure, and building formality with the set of all sequences.

Classifying examples from the initial set of sequences

The first example/set use category featured the students classifying sequences to develop, test, or refute a conjecture specific to their initial set of sequences. From our perspective, the students used their initial set of sequences as their universal set while they engaged in classifying; their diagram organized their initial set of sequences, and their discussion pertained to these specific sequences. An instance of this example/set use occurred during the fourth session in which the students were first asked to sort their initial set of sequences as bounded above and/or bounded below, or neither by placing their sequences in a corresponding set. In response, the students checked whether each sequence had an upper and/or lower bound (classifying). As they were classifying, Stella stated, "I feel like all of our a_n, $f(a_n)$, all those from that example, are going to be [bounded] both," referencing a set of sequences generated during a previous task. Here, we see that Stella noticed a pattern (searches for structure) to form a conjecture about the placement of some of their sequences from their initial set (conjecture development). From our perspective, she considered some of their initial set of sequences ("all of our a_n, $f(a_n)\ldots$") as a subset of their bounded below and bounded above sequences. Then, to confirm that the pattern held (belief confirmation), Stella and Lara continued to classify the sequences they generated based on whether there was an upper and lower bound.

After the students finished classifying their sequences as bounded above and/or bounded below, the teacher-researcher requested that the students add a place for increasing sequences.

Lara shared that it should be "somewhere in the middle" because there was a sequence that was increasing, bounded below, and bounded above. Stella then suggested that they place it "hereish" (see the blue category in Figure 2A[2]) because "some of the [only] bounded below are increasing." During this, Lara and Stella selected increasing sequences (attended to properties) and determined whether they were bounded below and/or above (searched for structure) to make conjectures about their initial set of sequences that connected increasing and the other properties (conjecture development). The teacher-researcher then responded, "Let's try it!... With the examples or the sequences that we have so far, we'll try to put them in one of these locations. And then if we find that we need to change that box, then we will do that." To respond to the teacher-researcher and to confirm their suspicion about their increasing sequences (belief confirmation), Lara and Stella considered each sequence to check if it was increasing and rearranged the sequences accordingly (classifying). See Figure 2B for their revised map. From our perspective, we see the students suggested that their increasing sequences were a subset of the union of (a) their bounded below and bounded above sequences and (b) the difference in the set of their bounded below sequences and the set of their bounded above sequences.

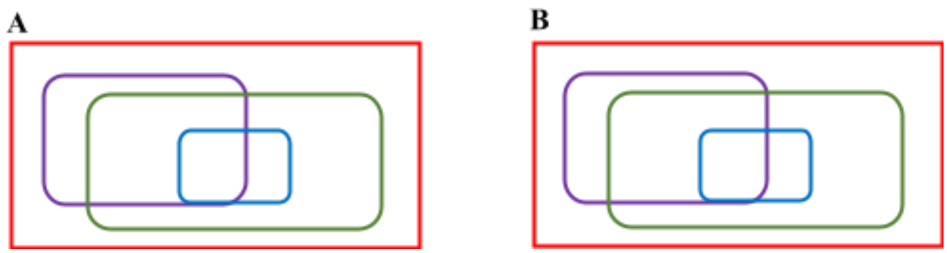
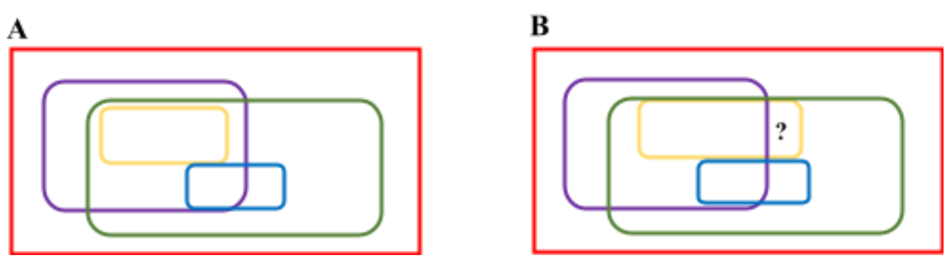


Figure 2. Location of the set of students' increasing sequences (blue set).

Seeking diversity and using lack of examples from an expanded initial set of sequences

The second category of example/set use featured the students seeking diversity in their initial set of examples and using the lack of examples to confirm their belief and understand why. From our perspective, the students expanded their universal set to include more sequences than their initial set while they sought diversity and used the lack of examples. Their discussion involved more than their initial set of sequences; in particular, they attempted to generate new sequences with different properties, and in doing so, their map transformed from organizing their initial set to potentially organizing additional sequences. An instance of this occurred later in the fourth session after Lara and Stella added a location for convergent sequences, which at the time they referred to as sequences "approaching a number" (see yellow set in Figure 3A). The teacher-researcher then asked, "Should [it] be stretched over?," changing the set's placement and adding a question mark to a new area to consider (see Figure 3B).



[2] We slightly altered these pictures from the students' version for clarity and because of space constraints. The students' version included more sequences with different representations (e.g., graphs).

Figure 3. Location of the set of sequences that approach a number.

To respond to the teacher-researcher and test the truth of the teacher-researcher's implied conjecture that there is a sequence that approaches a number, is unbounded above, and is bounded below, the students iteratively chose or created a new object to consider. In particular, they attempted to create a new sequence with the desired properties (seeking diversity, Figure 4A, Figure 4D, Figure 4E see below), selected ones on their map by paying attention to certain properties (attending to properties, Figure 4B), or created a new sequence based on a previous one (systematic variation, Figure 4C). With each new object, they disregarded it since it did not fit with a desired property (classify). Specifically, Figure 4A was bounded above since the intended domain was natural numbers, Figure 4B was not approaching a number, Figure 4C was bounded above, Figure 4D was bounded above and below, and Figure 4E was not approaching a number. After the students continued attempting to identify a sequence with the properties, Lara stated, "Maybe the answer is just no," meaning that there is not a sequence that is approaching a number, unbounded above, and bounded below. We interpret this as Lara reflecting on their inability to create an example with specific properties (using lack of examples) to confirm their initial placement of the converging set as non-overlapping with the difference of bounded above and bounded below (belief confirmation). From our view, the students suspected that the intersection of all convergent sequences, all bounded below sequences, and the complement of all bounded above sequences was empty (mutually exclusive).

Figure 4. Students' new objects.

After the teacher-researcher asked, "Can you think of a reason why we couldn't find a sequence?," the students began to understand why there was not a sequence that was approaching a number, unbounded above, and bounded below (understand why). To do so, Lara and Stella looked within the approaching sequences (attending to properties) and searched for a pattern, during which Lara noticed "no matter which section you take of the graph, it's bounded above and below" (searched for structure).

Attending to properties, searching for structure, and building formality with the set of all sequences

The third category of example/set use featured the students attending to properties, searching for structure, and building formality with the set of all sequences to convey general claims and general arguments. From our perspective, the students used the set of all sequences as their universal set (not just those depicted and organized on their map) while they attended to properties, searched for structure, and built formality. Students used examples and sets in this way during the fifth and sixth sessions, which began when the teacher-researcher asked the students to make conjectures based on their previous explorations. Stella responded, "I think it's true that a function cannot be both bounded above and increasing." After the teacher-researcher asked why not, Stella considered their increasing set and bounded above set (attended to properties) and seemed to notice that the increasing set intersected some of, but not all, the bounded above set (searched for structure), explaining, "So my thinking is, the blue increasing square does not have a category that's just bounded above. So, like increasing and just bounded above.” She then revised her conjecture (conjecture development), saying, "I guess you could say that it has to be bounded below to be increasing". From our perspective, Stella noticed that the set of increasing sequences was a subset of the set of bounded below sequences. After the teacher-researcher requested that they write this conjecture, the students began developing a formal representation in the form of a statement about what is the same across the increasing sequences (building formality), writing: “sequences that are increasing must be bounded below.” During the following session, the teacher-researcher asked for clarity about whether they considered this statement applicable for all or some increasing sequences. The students confirmed that they intended their conjectures to be about “all” increasing sequences rather than some of them or the specific examples in their map, and revised their conjecture to “all sequences that are increasing must be bounded below.”

The teacher-research then requested that they reflect on whether they thought the statement was true or knew it was true. Stella explained, "So I think we know this is true because all of the values are greater than or equal to the given number which would be… I guess it would have to be *x* equals one in this case.” Here, she noticed a pattern in that all the increasing sequences were bounded below by a value (searching for structure). She did this to understand what about the increasing sequences made them bounded below (understanding why). The teacher-researcher then questioned what she meant by $x=1$. To respond to the teacher-researcher and convey a general argument, Stella continued building formality to her argument revising her justification to : “all of these sequences [increasing sequences] must have a lower bound at a_n where $n = 1$.”

Discussion

In this study, we investigated how students' example and set use might evolve during their conjecturing and proving activity. Our findings suggest that students' example and set use can evolve over time, and specifically, we identified three categories of our participants’ example/set use: (1) classifying examples from the initial set of sequences, (2) seeking diversity and using lack of examples from an expanded initial set of sequences, and (3) attending to properties, searching for structure, and building formality with the set of all sequences.

Our findings support previous research that indicates students can productively use examples and sets in conjecturing and proving activities. Our study further contributes to this existing literature by shedding light on how students might use examples and sets in tandem during conjecturing and proving. In particular, the sets served as an organizational tool for students’ examples. These sets evolved from organizing their initial set of sequences to a map that depicted general claims of the set of all sequences.

We also gained some insight into guidance that supported the students' example/set use by attending to the *responding to the teacher-researcher* purpose. We found that the first type of example/set use was supported by requests for students to place sequences into their appropriate categories and/or add new categories. The second type was often supported by questioning whether the students' initial placement of the category should be revised by expanding it to include a new category. Moreover, questions like, "Can you think of a reason why we couldn't find a sequence?" further supported students to understand why. The last type of category was supported by requesting that students write their conjectures and then reflect on whether they thought they were true or knew they were true, and in the case that they thought it was true but were not certain, supporting them to further search for structure. In our future work we hope to further explore such guidance and example/set use.

References

Alcock, L., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*(2), 111–129. https://doi.org/10.1007/s10649-008-9149-x

Aricha-Metzer, I., & Zaslavsky, O. (2019). The nature of students' productive and non-productive example-use for proving. *Journal of Mathematical Behavior, 53,* 304–322. https://doi.org/10.1016/j.jmathb.2017.09.002

Cañadas, M. C., Deulofeu, J., Figueiras, L., Reid, D., & Yevdokimov, O. (2007). The conjecturing process: Perspectives in theory and implications in practice. *Journal of Teaching and Learning, 5*(1), 55–72. https://doi.org/10.22329/jtl.v5i1.82

Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal, 20*(1), 41–53. https://doi.org/10.1080/0141192940200105

Dawkins, P. C. (2017). On the importance of set-based meanings for categories and connectives in mathematical logic. *International Journal of Research in Undergraduate Mathematics Education, 3*(3), 496–522. https://doi.org/10.1007/s40753-017-0055-4

Dawkins, P. C., & Cook, J. P. (2017). Guiding reinvention of conventional tools of mathematical logic: Students' reasoning about mathematical disjunctions. *Educational Studies in Mathematics, 94*(3), 241–256. https://doi.org/10.1007/s10649-016-9722-7

Ellis, A. B., Ozgur, Z., Vinsonhaler, R., Dogan, M. F., Carolan, T., Lockwood, E., Lynch, A., Sabouri, P., Knuth, E., & Zaslavsky, O. (2019). Student thinking with examples: The criteria-affordances-purposes-strategies framework. *Journal of Mathematical Behavior, 53*, 263–283. https://doi.org/10.1016/j.jmathb.2017.06.003

Fiallo, J., & Gutiérrez, A. (2017). Analysis of the cognitive unity or rupture between conjecture and proof when learning to prove on a grade 10 trigonometry course. *Educational Studies in Mathematics, 96*(2), 145–167. https://doi.org/10.1007/s10649-017-9755-6

Freudenthal, H. (2005). *Revisiting mathematics education: China lectures* (Vol. 9). Springer Science & Business Media.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1–3), 111–129.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396–428. https://doi.org/10.2307/749651

Hub, A., & Dawkins, P. C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *Journal of Mathematical Behavior, 50*, 90–102. https://doi.org/10.1016/j.jmathb.2018.02.001

Iannone, P., Inglis, M., Mejía-Ramos, J. P., Simpson, A., & Weber, K. (2011). Does generating examples aid proof production? *Educational Studies in Mathematics, 77*(1), 1–14. https://doi.org/10.1007/s10649-011-9299-0

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*(1), 3–21. https://doi.org/10.1007/s10649-006-9059-8

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379–405. https://doi.org/10.2307/4149959

Komatsu, K., Jones, K., Ikeda, T., & Narazaki, A. (2017). Proof validation and modification in secondary school geometry. *Journal of Mathematical Behavior, 47*, 1–15. https://doi.org/10.1016/j.jmathb.2017.05.002
Lannin, J., Ellis, A., & Elliot, R. (2011). *Developing essential understanding of mathematical reasoning for teaching mathematics in grades Pre-K-8* (R. M. Zbiek, Ed.). National Council of Teachers of Mathematics.
Lin, M. L., & Wu, C. (2007). Uses of examples in geometric conjecturing. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 209–216).
Lockwood, E., Ellis, A. B., & Lynch, A. G. (2016). Mathematicians' example-related activity when exploring and proving conjectures. *International Journal of Research in Undergraduate Mathematics Education, 2*(2), 165–196. https://doi.org/10.1007/s40753-016-0025-2
Lockwood, E., & Purdy, B. (2019). Two undergraduate students' reinvention of the multiplication principle. *Journal for Research in Mathematics Education, 50*(3), 225–267. https://doi.org/10.5951/jresematheduc.50.3.0225
Lynch, A. G., & Lockwood, E. (2019). A comparison between mathematicians' and students' use of examples for conjecturing and proving. *Journal of Mathematical Behavior, 53*, 323–338. https://doi.org/10.1016/j.jmathb.2017.07.004
Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*(1), 41–51.
Ozgur, Z., Ellis, A. B., Vinsonhaler, R., Dogan, M. F., & Knuth, E. (2019). From examples to proof: Purposes, strategies, and affordances of example use. *Journal of Mathematical Behavior, 53*, 284–303. https://doi.org/10.1016/j.jmathb.2017.03.004
Pedemonte, B., & Buchbinder, O. (2011). Examining the role of examples in proving processes through a cognitive lens: The case of triangular numbers. *ZDM, 43*(2), 257–267. https://doi.org/10.1007/s11858-011-0311-z
Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics, 83*(3), 323–340. https://doi.org/10.1007/s10649-012-9459-x
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 160–182). Lawrence Erlbaum Associates.
Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*(3), 314–352. https://doi.org/10.5951/jresematheduc.40.3.0314
Swinyard, C., & Larsen, S. (2012). Coming to understand the formal definition of limit: Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education, 43*(4), 465–493. https://doi.org/10.5951/jresematheduc.43.4.0465
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. https://doi.org/10.1007/BF00305619
Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39*(4), 431–459. https://doi.org/10.5951/jresematheduc.39.4.0431