Students' Productive Techniques for Approaching Well-Definedness and Everywhere-Definedness

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Functions are critical in mathematics but have received limited attention at the advanced level. Research in advanced contexts has primarily focused on students' reasoning about specific types of functions (e.g., isomorphisms) but not on the function concept itself. In this paper, we explore students' productive techniques involving the definitive properties of function: well-defined and everywhere-defined. We found that the techniques students productively employed extended far beyond canonical procedures (like the vertical line test) and largely drew upon function meanings involving coordination of the domain, codomain, and rule, which previous research has highlighted the importance of but stopped short of directly investigating. Two contributions of this work include our focus on productive, successful techniques (rather than challenges and difficulties), and explicit focus on everywhere-defined (which has not received direct attention).

Keywords: function, well-defined, everywhere-defined, advanced mathematics

Functions are an essential concept in advanced mathematics underscoring ideas like binary operation, continuity, and homeomorphisms. We treat functions as having two defining properties (Even and Tirosh, 1995): "(1) they should be defined on every element in the domain, and (2) for each element in the domain there should be only one element (image) in the range; this condition is also known as univalence" (p. 4). We refer to the first property as everywhere-defined and the second as well-defined. These properties can serve to support or constrain student reasoning about abstract types of functions in later mathematics (Melhuish et al., 2020). For example, many students determine that functions like $f(x) = 1/x$ are not continuous because of their asymptotes (e.g., Takači et al., 2006). Students may not be considering that a function must be everywhere-defined on its domain. Similarly, in abstract algebra, well-definedness plays a crucial role as students work with functions on structures such as quotient groups whose elements are equivalence classes. In fact, Rupnow (2021) has documented that instructors may spend an entire lesson on these properties (along with onto and one-to-one).

The larger body of literature about student understanding of function primarily illustrates students' incomplete or non-generalizable meanings and techniques (e.g., Dorko, 2017; Even & Bruckheimer, 1998; Martínez-Planell & Trigueros Gaisman, 2012). Even and Tirosh (1995) proposed that the concept of well-definedness is often viewed in a rote and procedural way which is found to be superficial upon probing. Indeed, the most common manifestation—procedural or otherwise—of well-definedness at the K-12 level is the vertical line test or checking listed set of elements. The vertical line test can be a useful technique when accompanied with meanings of well-defined (e.g., Clement, 2001; Melhuish et al., 2020; Thomas, 2003); however, researchers have also noted that students over rely on the vertical line test and such techniques remain salient at the advanced level (e.g., Zandieh et al., 2017).

Beyond the vertical line test, researchers have also documented that students may not attend to function properties at all (e.g., Melhuish et al., 2020; Vinner, 1983). Thompson (1994) argued that the "predominant image evoked in students by the word 'function' is of two written expressions separated by an equal sign" (p. 5). Furthermore, introductory examples of function tend to draw on metaphors like the "function machine" (Tall & Bakar, 1992) which appeal to diagrams and tables where properties are readily identifiable by observation. Thus, even when students draw attention to the defining properties, it is not always in a way that is operable in terms of techniques for other contexts (e.g., Vinner 1983). We also note, despite the important role of everywhere-definedness (such as the continuity example above or in support of understanding homomorphism), few researchers have addressed this property explicitly, and we were not able to find any documented techniques. In this paper, we explore: What types of techniques do advanced mathematics students use to productively engage with well-defined and everywhere-defined properties when determining if a given relation is a function?

Theoretical Framing

We broadly take an approach aligned with Paoletti and colleagues' (2018) distinction between techniques and meaning. We share similar constructivist assumptions that students have developed their meanings for a particular concept based on an array of prior experiences. That is, the meanings students have for functions and the properties of well-defined and everywhere-defined reflect not an in-the-moment cognitive state, but rather more overarching understandings. In contrast, techniques refer to "student's words and observable activity as she addressed a single task" (p. 95). Students draw on their knowledge when engaging in problem-solving tasks, and techniques describe what they know-to (Mason & Spence, 1999) do in-the-moment. In our study, we were particularly interested in documenting the techniques used as students drew on their meanings for functions related to well-definedness and everywhere-definedness.

As noted in the introduction, it is essential that students move beyond a common meaning for functions as only a rule or formula. We suggest that function meaning requires coordination between the domain set, the codomain set, and rule that incorporates the two defining properties. If, when explaining why a proposed correspondence is or is not a function, students make explicit reference (in their language, gestures, or inscriptions) to (1) an element in the domain, (2) an element in the codomain, and (3) the rule, it would suggest that they are drawing on a meaning that has some coordination between these key components.

Methods

We conducted task-based clinical interviews (Clement, 2000; Goldin, 2000) with five students who (1) had already completed an advanced mathematics course (e.g., abstract algebra, cryptography, number theory) and (2) successfully approached basic "is this a function?" tasks. To identify such students, we reached out to several advanced mathematics courses at a large research university in the United States to ask for participants.

Data Collection and Task Design

We conducted two to three interviews with each student individually for an hour to an hour and a half each. Each interview was conducted and video-recorded over Zoom. The tasks were created with each category of non-examples from Uscanga and Cook (2022) being represented (sometimes in the modified form of an example). These categories are: *well-definedness – domain choice* (where an element in the domain can be represented in different equivalent ways and the symbolic rule maps these representations to distinct outputs), *well-definedness – codomain choice* (where the rule requires a choice of the outputs in the codomain, even though it does not invoke different equivalent representations), *everywhere-definedness – domain restriction* (where the symbolic rule maps an element in the domain to an output that is not contained in the proposed codomain), and *everywhere-definedness – domain expansion* (where the symbolic rule assigns to an element in the domain an output that is not contained in the proposed codomain but is contained in an easily accessible superset). The interviews were centered around four core tasks which asked students to identify if a proposed correspondence was an example or a non-example of a function. Additional tasks were designed for subsequent interviews to test and refine the hypotheses we established for each student, so they were adequately supported or refuted based on our observations.

Data Analysis

We engaged in both ongoing and retrospective analysis. Ongoing analysis occurred during and between sessions. This involved generating tasks and questions to test specific hypotheses about the students' reasoning. After data collection, we independently analyzed students' responses to five tasks. We identified excerpts from the transcripts that illustrated the techniques used by students to explore each task and coded these for definitional properties (well-defined or everywhere-defined), focus on domain, codomain, or rule, and the tool or approach utilized to solve the task. Codes were revised until agreement was reached and summaries of each student's approaches were created. This allowed us to compare techniques across students as well as across tasks. A second round of analysis involved grouping the data by type of property explored in order to identify the different techniques students used to productively engage with well-definedness and everywhere-definedness. Finally, we categorized these individual techniques into broader categories that described the general approaches students employed.

Results

Well-definedness

The well-defined property has been the subject of many empirical studies (e.g., Clement, 2001; Dorko, 2017; Even & Bruckheimer, 1998; Even & Tirosh, 1995). In this section, we highlight two overarching themes in students' activity: sameness and divergence. We begin by illustrating students drawing on prototypical examples in their open explorations, then consider the more sophisticated techniques used by students in more abstract settings.

Table 1. Students' techniques for engaging with well-definedness.

Technique	Focus	Method
Diagram or Table	Canonical Technique	Identify diverging outputs or repeats
Vertical Line Test	Canonical Technique	Identify multiple points along a same vertical line
Rule Ambiguity	Symbolic Rule	Identify if symbolic form assigns multiple outputs
Rule Equivalence	Symbolic Rule	Manipulate symbolic rule into known equivalent form
Attending to Function Notation Elements	Rule Focus, Domain Secondary	Determine if symbolic form of how an element is represented in input notation impacts output
Attending to Equivalence in the Domain	Domain Focus, Rule Secondary	Identify equivalence classes in the domain and explore their corresponding outputs
Attending to Equivalence in the Domain and Codomain	Domain, Rule, Codomain Focus	Identify equivalence classes in the domain and examine outputs via equivalence classes in the codomain

Canonical techniques. We suggest that the literature-based techniques (multiple outputs for one input identified via listing elements or vertical line test) might be deemed early canonical. Our advanced mathematics students drew on such examples when prompted to create functions and non-functions (see Figure 1). For example, Student B explained, “one of the guys from the domain goes to like, four different outputs, so that violates our condition.” Student C stated, for instance, “if we drew like a vertical line, we know there's a lot of points where, you know, our one input value has two different output values.” From the point-of-view of sameness and divergence, these techniques do not require identification of sameness, but rather rely on identifying divergence of outputs. They simplify the relationship between the domain, rule, and codomain to draw attention to inputs and outputs without substantial exploration of the surrounding sets.

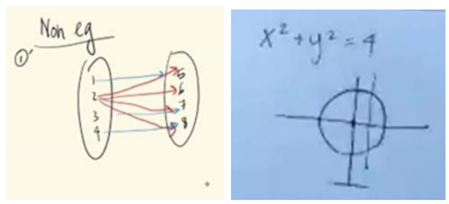

Figure 1. Left. Student B’s function diagram of an example and a non-example of a function. Right. Student C’s depiction of the vertical line test to establish a circle is not a function.

Attending to divergence. We begin by elaborating a technique that involves forefronting the symbolic rule which involves recognition of the potential for a rule to lead to divergence. Many of the students discussed this issue in relation to the correspondence $f: \mathbb{R} \rightarrow \mathbb{C}$ given by $f(x) = \sqrt{x}$. For example, Student A explained, “we have made a convention that if you just see a square root sign, you automatically decide that it’s the positive square root sign.” These students went on to provide a counterfactual that without this convention, it would not be a function, because, as noted by Student B, “our four could map to plus two or minus two. And then we don’t know which one we’re choosing, so that’s an ambiguity.” We note that the students are attending primarily to the symbolic rule but coordinating the role the rule plays diverging a single input to two outputs—drawing on the well-defined meaning. The students’ language also reflects sophistication in their meanings with attention to the role of convention and ambiguity.

Attending to equivalent functions. When exploring the rule $f: \mathbb{Q} \rightarrow \mathbb{Q}$ taking $\frac{a}{b}$ to $\frac{a+b}{b}$, several students worked with this unfamiliar rule with a lens of sameness: finding a familiar well-defined function. After some algebraic manipulation, Student B explained, “my assignment can be abbreviated as x going to $x + 1$, which is, which is a function.” We note that this approach is not explicating the coordination between sets and rule nor the definition of well-defined. However, we infer that the student using this technique likely had an awareness of that, as Student E stated, this function “is simply your input plus one” which is quite obviously well-defined. Thus, we suggest that manipulating the rule can be a productive technique if there is an obvious equivalent function.

Attending to function notation elements. Techniques that attend primarily to the rule are not sufficient in approaching a common area of concern for well-definedness: equivalence classes in the domain set. We identified two different ways that domain sameness was coordinated with rule divergence. The first attends primarily to the rule in terms of not just the outputs produced from an input, but also the notation of the input in function notation. For example, regarding the function $g: (0, \infty) \rightarrow \mathbb{R}$ given by $g(x) = \frac{1}{x}$, Student B explained that “given an x, it could be in the form a divided by b … But the formula, the assignment does not need that information.” This student is attending to elements of the domain (recognizing the set involved and potential representations), but then acknowledging that the function, as written, would not use that information. In contrast, for $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$, Student B explained, "the formula in this case, actually depends on the representation. Like, depends on how we have written the fraction." They go on to provide an example, $\frac{1}{2}$ and $\frac{2}{4}$ are the same, but do not give the same output (divergence). This student is drawing on meaning and coordination between the domain set ($\mathbb{Q}$) and the rule in order to determine if elements from the domain can have distinct representations and whether the rule, attending to function notation input, will act on those representations differently.

Attending to equivalence in the domain. Other techniques centered the domain to determine if an "equivalence" relation existed. This technique involved identifying equivalent elements (sameness) and then testing if the rule produced two outputs (divergence). Consider Student A's response to a modular arithmetic task ($f: \mathbb{Z}_3 \to \mathbb{Z}$ given by $[a]_3 \mapsto a$):

> Look, zero-three [writing $[0]_3$], I could write it a different way, right? … I could write this as six-three [writing $[6]_3$]. Those are the same element in $\mathbb{Z}_3$. … If I was gonna do six-three, your rule tells me that I have to map it to six. … the element zero-three maps to zero, six, three, nine … There's a lot of, a lot of outputs, and I'm only allowed to have one output.

This sort of exploration and attention to specific elements was found not just in the modular arithmetic tasks, but also the rational number tasks.

Everywhere-Definedness

We now transition to the everywhere-defined property which has received less attention in the literature. The results in this section are organized around three main themes that arose from students' activity: containment, existence, and set operation. We begin with a students' prototypical example from their open exploration and then explore more sophisticated techniques used by the different students throughout the interviews.

Table 2. Students' techniques for engaging with well-definedness.

Technique	Focus	Method
Diagram	Canonical Technique	Identify inputs that do not have corresponding outputs
Attending to Output Containment	Codomain and Rule Focus	Determine if outputs assigned by the symbolic rule are contained in the proposed codomain
Attending to Output Existence	Domain and Codomain Focus, Rule Secondary	Determine if symbolic rule assigns an output to every input
Manipulating Sets via Extending	Domain and Codomain Focus, Rule Secondary	Operate on domain or codomain by extending the set and focusing on a superset
Manipulating Sets via Subsetting	Domain and Codomain Focus, Rule Secondary	Operate on domain or codomain by subsetting the set and focusing on a subset
Manipulating Sets via Partitioning	Domain and Rule Focus, Codomain Secondary	Subdivide domain into sets where the symbolic rule causes known problem areas

Canonical technique. The students rarely evoked everywhere-defined while generating examples and non-examples. Only one student attended to it when asked to provide an example of a non-function. We deemed their approach canonical as it is the type of example students might be used to from K-12 settings. As seen below (Figure 2), the student drew a function diagram where one element in the domain had no corresponding output in the codomain. They explained:

> I was like “oh, well, could I have no outputs? … Would that be, a part, would that break it being a function?” Well, that wouldn't even count because a relation requires that each input has at least one output … So, I could make it not a relation by sticking this F in here, and it has no arrow. Well, then it’s not a function because it’s not a relation.

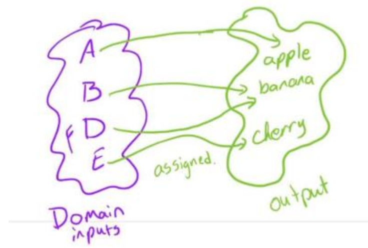

Figure 2. Student A’s function diagram which fails the everywhere-defined property.

Attending to output containment. Several students attended to whether the outputs assigned by the rule were contained in the proposed codomain. They viewed the rule as assigning an output to each input and focused on whether the resulting outputs were contained in the codomain specified. For instance, when dealing with the non-example $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$, Student A explained, “this relation has its own meaning, and its meaning, says, something about, this $\mathbb{N}$ right here, having, to include the answer that the rule would go to.” In more detail, they noted:

> What happens when I plug in negative two? Right? Well then the rule, says that, I have to go to negative two cubed … that has to be negative eight, and negative eight does not live in $\mathbb{N}$ … So I can’t say if it’s a function because it’s a nonsensical thing … Negative two lives here, look it’s right there, and it’s telling me to go to negative eight, but negative eight, I can’t, I can’t go. It won’t let me. This, this won’t let me go there.”

This reflects a two-stage process, attending to the rule and then coordinating with the codomain.

Attending to output existence. A slightly different technique focused on the correspondence not assigning outputs to certain inputs. For example, Student B used this technique to explain why $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$ was not an example of a function:

> So the first thing that we need to have is, we have to have a relation between stuff in the range, the stuff in the domain and the stuff in the range. But, so if I tried to write down an ordered pair here, I will not have anything to write for minus one. So, that, we can’t even get an ordered pair. So we can’t even get a relation for minus one.

This student argued that there is no output assigned to -1. The attention is on the domain and codomain and the rule either relates the two sets or does not.

Manipulating sets via extending and subsetting. Two other techniques that arose involve operating on the domain or codomain by extending the set or subsetting it. For instance, Student E subsetted the codomain when working with the square root function and explained how they restricted their focus on a subset, “So if it was negative, it would result in a complex number, but when you’re considering the function is mapping from reals to reals, then you have to restrict the domain. But it’s still a function.” They continued, “It’s just your domain, the y values don’t exist for this particular field that we're considering. But you defined the mapping as from reals to the complex numbers, which means that your domain is unrestricted here."

We also observed students extending the domain, as Student E did when discussing the function $g\colon (0,\infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$, "since you restricted your domain, this is a function … if you would have said that you were just mapping from reals … x cannot equal to zero … you do not have a valid y value for that." The student extended the domain to explore the issues that arise with the correspondence and notes that the domain is not extended in this case so there are no issues. In this technique, the students' focus is on a superset of the specified domain or codomain, whereas in the first technique the student focuses on a subset of the proposed domain or codomain.

Manipulating sets via partitioning. The other related technique involved subdividing the domain as a result of a known problem area related to the symbolic form of the rule. Students then further made arguments about containment of the outputs. For example, Student C noted about the x^3 non-example:

> So our two sets here is the integers and the natural numbers. And so this, function that we wrote here, the x cubed, you know, there are some things that "okay, yeah we can put some numbers in for x and we can get some natural numbers, that's fine." But there's also some numbers, you know, our negative numbers, if we put negative numbers in here, we're not getting output values in the natural numbers.

Here, the student is subdividing the domain into the positive and the negative integers since negative integers could have issues with this rule. By exploring the places where possible issues might arise with the symbolic rule, students were able to determine whether the provided correspondences were or were not functions.

Discussion

In this paper, we described different types of techniques that advanced mathematics students used to productively engage with the properties of well-defined and everywhere-defined when determining if a given correspondence is a function. We found that the students in the study used techniques that extended far beyond the canonical techniques that rely primarily on observing irregularities (multiple outputs or lack of outputs). For well-defined, students used ideas of convention and ambiguity to attend to issues with symbolic rules and naming conventions, and sameness and divergences to consider equivalence classes and representations in the codomain and domain sets. For everywhere-defined, students used ideas of containment and existence to consider implications of certain rules and operated on sets to consider extensions and subsets that may complicate the relationship with the rule. This is a key contribution because while researchers had previously examined various facets of reasoning productively with functions in mathematics in general (e.g., Oehrtman et al., 2008; Tall & Vinner, 1981) and in advanced mathematics in particular (e.g., Melhuish et al., 2020; Zandieh, et al., 2017), investigation into well-definedness and everywhere-definedness has been rather limited.

These results can provide a foundation for additional research that could attend to ways students might develop these techniques. Future research can elaborate how techniques like the ones identified here might influence students' reasoning about more advanced concepts within the context of abstract mathematics where these properties have crucial roles in formal proofs.

References

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-590). Lawrence Erlbaum Associates, Inc.

Clement, L. L. (2001). What do students really know about functions? *Mathematics Teacher, 94*(9), 745-748.

Dorko, A. (2017). Generalising univalence from single to multivariable settings: The case of Kyle. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 562-569).

Even, R. & Bruckheimer, M. (1998). Univalence: A critical or non-critical characteristic of functions? *For the Learning of Mathematics, 18*(3), 30-32.

Even, R. & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics, 29*, 1-20. https://doi.org/10.1007/BF01273897

Goldin, G. A. (2000). A scientific perspective on task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-546). Lawrence Erlbaum Associates, Inc.

Martínez-Planell, R. & Trigueros Gaisman, M. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics, 81*, 365-384. https://doi.org/10.1007/s10649-012-9408-8

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Forms of mathematical knowledge: Learning and teaching with understanding*, 135-161.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *The Journal of Mathematical Behavior, 60*, 1-16. https://doi.org/10.1016/j.jmathb.2020.100806

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Mathematical Association of America.

Paoletti, T., Stevens, I. E., Hobson, N. L. F., Moore, K. C., & LaForest, K. R. (2018). Inverse function: Pre-service teachers' techniques and meanings. *Educational Studies in Mathematics, 97*, 93-109. https://doi.org/10.1007/s10649-017-9787-y

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructor's descriptions for themselves and when teaching. *The Journal of Mathematical Behavior, 62*, 1-14. https://doi.org/10.1016/j.jmathb.2021.100867

Takači, D., Pešić, D., & Tatar, J. (2006). On the continuity of functions. International Journal of Mathematical Education in Science and Technology, *37*(7), 783-791.

Tall, D., & Bakar, M. (1992). Students' mental prototypes for functions and graphs. *International Journal of Mathematical Education in Science and Technology, 23*(1), 39-50. https://doi.org/10.1080/0020739920230105

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Thomas, M. (2003). The role of representation in teacher understanding of function. In N. A. Pateman, B. J. Dougherty, & J.T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of*

PME and PMENA (Vol. 4, pp. 291-298). Center for Research and Development Group, University of Hawaii.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *CBMS Issues in Mathematics Education: Research in Collegiate Mathematics Education, I* (Vol. 4, pp. 21-44). American Mathematical Society.

Uscanga, R., & Cook, J. P. (2022). Analyzing the structure of the non-examples in the instructional example space for function in abstract algebra. *International Journal of Research in Undergraduate Mathematics Education*, 1-27. https://doi.org/10.1007/s40753-022-00166-z

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14*(3), 293-305. https://doi.org/10.1080/0020739830140305

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics, 95*, 21-38. https://doi.org/10.1007/s10649-016-9737-0