The Meta-Narratives about Function Conveyed by
a Commonly Used Multivariable Calculus Textbook

Brady A. Tyburski
Michigan State University

Function is a unifying, cross-curricular theme that plays a central role in nearly every subdiscipline of mathematics. Yet, this cultural meta-narrative that is widely accepted by mathematicians is not one students readily adopt. In this report, I share a preliminary analysis of the stories told about three different types of multivariable functions in a commonly used calculus textbook. In doing so, I juxtapose these stories and consider how this textbook—a cultural artifact of the discipline of mathematics—might support or hinder the transmittal of desirable cultural meta-narratives about the role of function. Ultimately, I find that real-valued functions are consistently positioned as functions, while the function properties of both types of vector-valued functions are de-emphasized. As only one of the three function types are positioned as functions, this suggests that the stories in the textbook could hinder students' adoption of the meta-narrative of function as a unifying mathematical theme.

Keywords: Function, Multivariable Calculus, Curriculum as Story, Textbook Analysis.

Function is a crucial recurring theme across the story of the K-16 mathematics curriculum. (e.g., CCSSM, 2010; Zorn, 2015). Some mathematics education researchers even go as far as to contend that the concept of function is "the single most important mathematical concept studied from kindergarten to graduate school" (Harel & Dubinsky, 1992, p. vii). Mathematicians have also spoken about the centrality of the function concept. For instance, Gowers et al. (2008) stated how "One of the most basic activities of mathematics is to take a mathematical object and transform it into another one" (p. 10). In this sense, function is a unifying, cross-curricular theme that occurs across nearly every subdiscipline of mathematics.

This cultural meta-narrative about the role of function has endured for decades. Yet, this is not a narrative that students readily adopt, as a large body of research suggests (e.g., Martínez-Planell & Trigueros, 2021; Melhuish, 2020; Zandieh et al., 2017). While there are several reasons for this, in this preliminary report, I examine the role that the stories told in a commonly adopted calculus textbook (Stewart et al., 2021)—a cultural artifact of the discipline of mathematics (Plut & Plesic, 2003)—might play in transmitting or failing to transmit this cultural meta-narrative to students. Specifically, I examine the stories told about the three different types of multivariable functions featured in multivariable calculus (MVC)—parametric, vector-valued functions; multivariable, real-valued functions; and vector fields—in the chapter they are first introduced. In this arts-based textbook analysis, I lean into the metaphor of curriculum as story (Dietiker, 2015) and treat each of these function types as mathematical characters. I investigate two questions: *(1) How are these characters portrayed in the chapters they are introduced and how do these portrayals compare with one another? (2) What meta-narratives about function(s) are conveyed collectively by these character introductions?*

Background

MVC textbook analyses that focus on the concept of function are admittedly limited. Notably, McGee et al. (2015) observed that in commonly used calculus textbooks, explicit conversations linking representations of single-variable and multivariable functions are virtually nonexistent, even though students may not spontaneously draw these connections on their own (Martínez-Planell & Gaisman, 2012). More generally, Harel (2021) noted that traditional MVC textbooks tend to introduce key concepts without proper motivation and introduce computational shortcuts prematurely. While other analyses attend to the meta-narratives conveyed by textbooks, they tend to be with an eye to a particular topic, like line integrals (Dray & Manogue, 2023).

Textbooks as Cultural Artifacts & Curriculum as Story

In undergraduate education, many instructors use textbooks as a primary curricular guide (Fraser & Bosanquet, 2006). So, even though mathematics students do not read them cover to cover (Weinberg et al., 2012), the stories conveyed (or not) in textbooks play a powerful role in the reproduction of mathematical culture and specifically the meta-narratives that are valued by the discipline (Plut & Plesic, 2003). By a *meta-narrative*, I mean a "cultural narrative schema which orders and explains knowledge and experience" (Stephens & McCallum, 1998, p. 6). In other words, meta-narratives are narratives that recur (implicitly or explicitly) across cultural artifacts which individuals subsequently leverage to frame and explain their past and subsequent experiences. For example, the meta-narrative that "good always conquers evil" is common across children's books. That said, as Brown (2022) demonstrated, the meta-narratives perpetuated by undergraduate mathematics textbooks do not always align with those held by members of the discipline of mathematics. In these cases, textbooks might actually convey messages that serve a counterproductive role toward enculturating students into the discipline.

I adopt the perspective that mathematics curriculum (and therefore textbooks) can be conceptualized as a story (Dietiker, 2015; Gadanidis & Hoogland, 2003). Like a story, curricula features (mathematical) characters inhabiting settings and engaging in actions that constitute the plot. A curriculum also features literary themes and morals in the same way a good story might. As textbooks are used for enculturating students into the discipline of mathematics, the stories contained within them should not merely be an afterthought. After all, the stories we are told and then re-tell influence how we organize our experiences (Clark & Rossiter, 2008). Indeed, the power of story has been acknowledged repeatedly by mathematicians (Doxiadis & Mazur, 2012) as well as mathematics educators (Burton, 1999), but analyses of the stories conveyed in textbooks from a literary and aesthetic angle remain rare (e.g., Dietiker & Richman, 2021).

I employ Dietiker's (2015) framework for analyzing mathematical stories based on the narratological work of Bal (2017). Specifically, I attend to the characters, action, settings, and plot. Mathematical characters are those concepts that get objectified in the text, including functions, numbers, etc. They are often imbued with *character traits* (e.g., a function may be 1-1, even, etc.). I attend specifically to which mathematical characters are positioned as *protagonists* and *side characters*, as well as the *relationships* between characters. Mathematical action refers to moments where a new character is created or introduced as well as when characters are manipulated mathematically (e.g., the characters "2" and "3," might be added to form a new character, "5"). The setting of mathematical stories includes different representations (tables, graphs, etc.) as well as physical contexts. Finally, the mathematical plot includes a reader's aesthetic reactions to the unfolding story as they attempt to discern its structure.

Data & Methods

I analyzed the character introductions of three different multivariable function types in Stewart et al.'s (2021) *Calculus*, a commonly adopted textbook in U.S. undergraduate classrooms for teaching the calculus series (Mesa, 2010; Mkhatshwa, 2022). By "character introduction", I mean the first section within the unit in which each function type is introduced (e.g., 13.1). Hereafter, I refer to these sections as "chapters" as a reminder that I am conceptualizing the textbook as a literary story. The three chapters analyzed were: (1) 13.1: Vector Functions and Space Curves, within the Vector Functions unit, in which parametric, vector-valued functions are introduced; (2) 14.1: Functions of Several Variables, within the Partial Derivatives unit, in which multivariable, real-valued functions are introduced; and (3) 16.1: Vector Fields, within the Vector Calculus unit, in which vector fields are introduced. I also included the paragraph overview of each unit because they immediately precede each chapter and briefly introduce each function type.

Methods

I first read the story of each chapter three times while making marginal notes in the same way someone might while trying to interpret literary fiction. Following an arts-based approach (Leavy, 2018; McNiff, 2018), I did not restrict the structure or modality of these notes. Instead, I focused on recording my thoughts, feelings, and observations in whatever modality was most appropriate. Sometimes, I would write a comment or pose a question; other times, I would draw a sketch or write a quick poem to artistically convey my reactions as a reader.

To add focus to my analysis, I attended primarily to the story elements outlined in Dietiker's (2015) framework (character, action, setting, plot). However, I did not shy away from attending to other dimensions that were salient in my reading of each story, especially those related to my aesthetic interpretations of how the story was told or the overall moral of each story. My intent was to lean into my subjectivity (Tremaine & Hagman, 2023) and unique positionality as a reader who is a disciplinary expert and has taught MVC multiple times. This involved engaging my full senses and emotions: I did not restrict myself to purely logical analysis of structure, mathematical content, etc. I allowed myself to wonder about what came next, why the textbook was or was not introducing certain elements, and to make aesthetic judgements.

Following this open-ended analysis, I re-read and reflected across my marginal notes and conducted secondary analyses to follow up on recurring themes and questions that had appeared across all the character introductions. In this report, I share the preliminary results from two such analyses: (1) a "nickname analysis" of the changing names and phrases used to refer to each character (e.g., function, equation, curve) paired with (2) a plot structure analysis of each story.

Story Analyses

Parametric, Vector-Valued Functions (PVVFs)

As soon as the unit introduction, the purpose of introducing PVVFs is clearly signposted: "We now study functions whose values are vectors because such functions are needed to describe curves and surfaces in space" (p. 927). While PVVFs are introduced formally as "a function whose domain is a set of real numbers and whose range is a set of vectors" (p. 928), they are soon relegated to a side character used primarily to introduce the main character of space curves. In fact, once the relationship between these characters has been established, the word function only appears two more times. Additionally, across the entire chapter, "function" is mentioned only about half as frequently as "curves" and about the same number of times as (parametric) "equation". The set theoretic functional properties (e.g., domain and range) of PVVFs are effectively backgrounded in favor of sketching curves and writing parametric equations in the subsequent examples.

Ultimately, the character of PVVFs appears to have been introduced as a technicality: even before space curves are introduced, the (unboxed) function definition is followed quickly by the introduction of component functions. After this, the primary action involving PVVFs is to quickly reduce them to their component single-variable functions for further computation. This backgrounding of PVVFs as functions sends a strong message about the defining traits of such functions: they decompose into multiple other single-variable functions and *these* functions are the most important.

Multivariable, Real-Valued Functions (MRVFs)

The character of MRVFs, on the other hand, is treated as a function from beginning to end. Unlike both other chapters analyzed, this one starts by outlining four points of view for studying MRVFs: verbally, numerically, algebraically, and visually (by a graph or level curves). This phrasing positions the words, tables, formulas, graphs, and level curves as in service of MRVFs, rather than the other way around (as was the case with PVVFs and geometric space curves). Additionally, this is the same framing used to introduce the function concept in the first (single-variable) chapter of the textbook, further affirming that MRVFs are, by their nature, functions.

Unlike with PVVFs, the formal set-theoretic definition of MRVFs of two variables is boxed and even "domain" and "range" are bolded, further emphasizing their functional characteristics. Next, the independent and dependent variables of a MRVF are defined and followed immediately by an aside to the reader that draws a comparison between single- and multivariable RVFs: "Compare this with the notation y = f(x) for functions of a single variable" (p. 927). Interleaved throughout the remaining pages are recurring explicit references to the domain and range in text and in the examples. Finally, the chapter ends by introducing the formal definition of MRVFs of three or more variables, returning to highlight that these characters are functions.

"Function" is far and away the most common name used to refer to MRVFs (87 instances). This use of this name does not subside, even as "level surfaces" (51 instances) and geometric names (such as "graph" or "surface", 44 instances) are introduced as alternatives.

Vector Fields

The caption for the image featured in the vector calculus unit introduction immediately states how "vector fields can be used to model such diverse phenomena as gravity, electricity and magnetism, and fluid flow" (p. 1161). Nearby, the first sentence of the main text reads, "In this chapter we study the calculus of vector fields (These are functions that assign vectors to points in space)" (p. 1161). The use of parentheses to convey the technical details persists throughout this paragraph and feels almost like the narrator is whispering to the reader, implicitly conveying that perhaps the function definition of vector fields is a sidenote.

The emphasis on modeling realistic phenomena carries into the beginning of the chapter introducing vector fields, which features nearly a full page of four example velocity fields depicted visually alongside written interpretations of what meaning the plotted vectors convey. Afterwards, the general definition of vector field is introduced (unboxed) as "a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3)" (p. 1163) followed by separate boxed definitions of a vector field on $\mathbb{R}^2$ and on $\mathbb{R}^3$. Curiously, the language of domain and range is less explicit in the boxed definitions: "Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field** on $\mathbb{R}^2$ is a function **F** that assigns to each point (x,y) in D a two-dimensional vector **F**(x,y)" (p. 1163). After each definition, component functions are introduced, with a brief mention that the reader should already be familiar with these characters from reading about them alongside PVVFs.

After two examples concerning how to sketch some specific vector fields, the story shifts primarily to what can best be described as extended vignettes of notable types of vector fields (e.g., a gravitational field, electric field, gradient fields, etc.). Aside from using function and vector notation as a symbolic necessity, once technology is introduced as a way of plotting vector fields, the functional aspects of vector fields go almost entirely unmentioned. Indeed, there are only 5 explicit references to vector fields as "functions" across this entire chapter, most of which occur near the formal definitions.

Discussion

These story analyses emphasize key differences between the character introductions of these three types of multivariable functions in one commonly used MVC textbook. While real-valued functions are clearly positioned as a main character and repeatedly portrayed as a type of function, the same cannot be said about both types of vector-valued functions. The difference is most extreme in the case of parametric, vector-valued functions, which play the role of a secondary character meant to introduce space curves and whose function character traits are quickly backgrounded. Vector fields remain the main character of their story; however, their functional character traits are mostly de-emphasized across a series of vignettes that focus on introducing and interpreting the physical meanings of specific vector fields (e.g., velocity fields, gravitational fields, etc.). Considered collectively, these stories perpetuate the meta-narrative that the most important type of multivariable functions in MVC are the real-valued ones. Vector-valued ones are also functions—at least formally—but this is portrayed as a mere technicality and not a fact used frequently in practice. Readers (our students) are more likely to come away with alternative messages about vector-valued functions that are foregrounded in these stories, such as how readily they can be decomposed into component functions ("the truly important functions") for the purposes of further computation. Or, in the case of vector fields, the importance of physical interpretation of vector outputs.

Whether these alternative messages are consistent with the curricular goals of MVC is beyond the scope of this preliminary report. However, it is worth noting that the dominance of such messages may hinder students from picking up on the cultural importance of function as a unifying, cross-curricular theme that includes both RVFs and VVFs. Rather, readers may be led to believe the meta-narrative that function is an unnecessary boondoggle, "extra", or formalism that only plays a passing role in the overarching story of MVC and, more generally, mathematics. Already, research suggests MVC students often need support in making sense of MRVFs as a generalization of single-variable real-valued functions. The same may well be true for making sense of vector-valued functions as multivariable functions, *especially* if there are even fewer textbook supports for making this generalization.

There is emerging empirical evidence suggesting students benefit from reasoning about different function types as instantiations of the same overarching function concept (Melhuish et al., 2020; Zandieh et al., 2017); however, none of these studies are in the context of MVC. I hypothesize that MVC students may similarly benefit from recognizing PVVFs, MRVFs, and vector fields as functions, as this would better position them to recognize structural similarity across the different types of calculus required for each function type (e.g., the Chop, Multiply, Add conceptual pattern for integration, Dray & Manogue, 2023). Simultaneously, I recognize that while an overarching story for MVC centered on function may support students' enculturation into the discipline of mathematics, it may have the opposite effect on students' enculturation into other STEM disciplines, given the differing ways that scientists and mathematicians conceptualize functions (e.g., Dray & Manogue, 2004). This suggests further cross-disciplinary research collaboration is needed to ensure that our MVC curricula feature coherent stories which set up our STEM students to be successful, regardless of their discipline.

References

Bal, M. (2017). *Narratology: Introduction to the theory of narrative* (4th ed.). University of Toronto Press.

Brown, S. A. (2022). The staging of proof by contradiction in texts: An exploration of disciplinary artifacts. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 752–759).

Burton, L. (1999). The implications of a narrative approach to the learning of mathematics. In B. Leone (Ed.), *Learning from mathematics: From hierarchies to networks* (pp. 21–35). Falmer Press.

Clark, M. C., & Rossiter, M. (2008). Narrative learning in adulthood. *New Directions for Adult and Continuing Education*, 119, 61–70. https://doi.org/10.1002/ace.306

Dietiker, L. (2015). Mathematical story: A metaphor for mathematics curriculum. *Educational Studies in Mathematics*, 90, 285–302. https://doi.org/10.1007/s10649-015-9627-x

Dietiker, L., & Richman, A. S. (2021). How textbooks can promote inquiry: Using a narrative framework to investigate the design of mathematical content in a lesson. *Journal for Research in Mathematics Education*, 52(3), 301–331. https://doi.org/10.5951/jresematheduc-2020-0318

Doxiadis, A., & Mazur, B. (Eds.). (2012). *Circles disturbed: The interplay of mathematics and narrative*. Princeton University Press.

Dray, T., & Manogue, C. A. (2004). Bridging the gap between mathematics and the physical sciences. In D. Smith & E. Swanson (Eds.), *Preparing future science and mathematics teachers* (pp. 39–41). https://bridge.math.oregonstate.edu/papers/bridge.pdf

Dray, T., & Manogue, C. A. (2023). Vector line integrals in mathematics and physics. *International Journal of Research in Undergraduate Mathematics Education*, 9, 95–120. https://doi.org/10.1007/s40753-022-00206-8

Fraser, S. P., & Bosanquet, A. M. (2006). The curriculum? That's just a unit outline, isn't it? *Studies in Higher Education*, 31(3), 269–284. https://doi.org/10.1080/03075070600680521

Gadanidis, G., & Hoogland, C. (2003). The aesthetic in mathematics as story. *Canadian Journal of Science, Mathematics, and Technology Education*, 3(4), 487–498. https://doi.org/10.1080/14926150309556584

Gowers, T., Barrow-Green, J., & Leader, I. (2008). Introduction. In T. Gowers, J. Barrow-Green, & I. Leader (Eds.), *The Princeton companion to mathematics* (pp. 1–76). Princeton University Press.

Harel, G. (2021). The learning and teaching of multivariable calculus: A DNR perspective. *ZDM*, 53, 709–721. https://doi.org/10.1007/s11858-021-01223-8

Harel, G., & Dubinsky (Eds.). (1992). The concept of function: Aspects of epistemology and pedagogy. Mathematical Association of America.

Leavy, P. (2018). Introduction to arts-based research. In P. Leavy (Ed.), *Handbook of arts-based research* (pp. 3–21). The Guildford Press.

Martínez-Planell, R., & Gaisman, M. T. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics*, 81(3), 365–384. https://doi.org/10.1007/s10649-012-9408-8

Martínez-Planell, R., & Trigueros, M. (2021). Multivariable calculus results in different countries. *ZDM*, 53, 695–707. https://doi.org/10.1007/s11858-021-01233-6

McGee, D., Moore-Russo, D., & Martinez-Planell, R. (2015). Making implicit multivariable calculus representations explicit: A clinical study. *Problems, Resources, and Issues in*

Mathematics Undergraduate Studies, 25(6), 529–541. https://doi.org/10.1080/10511970.2015.1038858

McNiff, S. (2018). Philosophical and practical foundations of artistic inquiry: Creating paradigms, methods, and presentations based in art. In P. Leavy (Ed.), *Handbook of Arts-Based Research* (pp. 22–36). Guilford Press.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior*, 60, 1–16. https://doi.org/10.1016/j.jmathb.2020.100806

Mesa, V. (2010). Strategies for controlling the work in mathematics textbooks for introductory calculus. *CBMS Issues in Mathematics Education*, 16, 235–261. https://doi.org/10.1090/cbmath/016/09

Mkhatshwa, T. P. (2022). Quantitative and covariational reasoning opportunities provided by calculus textbooks: The case of the derivative. *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2022.2129497

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common core state standards for mathematics.

Plut, D., & Pesic, J. (2003). Toward a Vygotskian theory of textbook. *Psihologija*, 36(4), 501–515. https://doi.org/10.2298/PSI0304501P

Stephens, J., & McCallum, R. (1998). *Retelling stories, framing culture: Traditional stories and metanarratives in children's literature*. Garland Publishing.

Stewart, J., Clegg, D., & Saleem, W. (2020). *Calculus* (9th ed.). Cengage Learning.

Tremaine, R., & Hagman, J. E. (2023). Creativity and subjectivity: Two researchers' poetic re-presentations of critical student perspectives of mathematical sciences. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1117–1124).

Weinberg, A., Wiesner, E., Benesh, B., & Boester, T. (2012). Undergraduate students' self-reported use of mathematics textbooks. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 22(2), 152–175. https://doi.org/10.1080/10511970.2010.509336

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics*, 95(1), 21–38. https://doi.org/10.1007/s10649-016-9737-0

Zorn, P. (2015). 2015 CUPM curriculum guide to majors in the mathematical sciences. Mathematical Association of America. https://www.maa.org/sites/default/files/CUPM%20Guide.pdf