Physics Students' Challenges Coordinating Multiplication with the Definite Integral Concept

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Integration plays a significant role in applied problems, including those set in a wide range of physical phenomena. Past research has shown that students often experience difficulties applying calculus knowledge to physics, engineering, and other subjects. More specifically, recent research has identified students' difficulties constructing the "product layer," $f(x) \cdot dx$*, as a core part of the definite integral. In this study, we extend research on students' understandings of the definite integral by focusing on how they reason about multiplication with quantities, physical units, and the construction of the "product layer."*

Keywords: Calculus, Integration, Product layer, Knowledge-in-Pieces

Introduction and Literature Review

Research on the teaching and learning of calculus has focused primarily on limits and differentiation (e.g., Larsen et al., 2017; Rasmussen et al., 2014), but there is an emerging body of literature on integration. Among other things, research on integration has observed that the "product layer," $f(x) \cdot dx$, can pose particular challenges for students. This same literature has not examined closely how students coordinate reasoning about multiplication with quantities and reasoning about integrals. This gap limits our ability to help students develop more robust understandings of the "product layer" and of the definite integral.

We emphasize three themes in the literature on definite integrals. First, students often interpret the definite integral as "area under the curve," as a "derivative/antiderivative" relationship, or as "adding up pieces/multiplicatively-based summation" (e.g., Jones, 2015a, 2015b; Ely & Jones, 2023; Jones & Ely, 2023). These researchers have argued that first two interpretations provide limited help to students when making sense of the definite integral in problem situations and that the third interpretation is more helpful when applying the integral to problems in physics, engineering, and other fields.

Second, students often have difficulties interpreting the "product layer," $f(x) \cdot dx$, as expressing multiplication (e.g., Ely, 2017; Jones, 2013; Sealey, 2014). Sealey (2014) presented the most direct framework for characterizing student understanding of Riemann sums and definite integrals. This framework consists of five layers: orienting, product, summation, limit, function. She reported that conceptualizing the "product layer" was the most complex part of constructing definite integrals to model problem situations. She concluded that students had trouble not with calculations but with "understanding what is being multiplied together and what quantity is produced from that multiplication" (p. 240).

Third, students often have difficulties with modeling quantities such as distance and work with rectangular areas (e.g., Thompson et al., 2013; Nguyen & Rebello, 2011; Christensen & Thompson, 2010). Thompson and Silverman (2008) pointed out that for students to perceive the area under a curve as representing a quantity other than area (e.g., velocity, force), it is important for students to consider the quantity being accumulated as a sum of infinitesimal elements formed multiplicatively. As a result, they proposed a model which emphasizes two "layers" of integration: the multiplicative layer when the bits are formed and the summation layer when the bits are accumulated.

In the present study, we examined how students construct the "product layer," which can be used to relate quantities and to build new quantities in the given problem situation. In addition, we focused on knowledge resources students used in the process of forming the "product layer." We asked the following two research questions:

1. *How do students use/construct the "product layer" to relate quantities and form new quantities in problem situations?*
2. *What knowledge resources do students apply when using/constructing the "product layer" to relate quantities and form new quantities in problem situations?*

Theoretical Framework

In this section we present the theoretical framework which best spans the results obtained in the study. It combines mathematical structure (as perceived by experts) with cognitive components evidenced by students.

Vergnaud (1983, 1988) analyzed mathematical structures for multiplication with quantities and distinguished two subtypes of multiplication situations. Each component in a product is associated with a *measure space* and the relation between the components is either an *isomorphism-of-measure-spaces* situation *(I-O-M)* or a *product-of-measure-spaces* situation *(P-O-M)*. The former means that the product structure consists of a simple direct proportion between two measure spaces M_1 and M_2 (1983, p. 129). An example is the product structure which has the form $\frac{meters}{second} \cdot seconds$. The latter means the product structure consists of the Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3 (1983, p. 134). An example is the product structure which has the form $area\ (cm^2) \cdot height\ (cm)$. These subcategorization of problem situations helps us delineate definite integral problems which possess two different product structures.

For the cognitive component, we draw from the knowledge-in-pieces epistemological perspective. The perspective was first developed in science education research on conceptual change (e.g., diSessa, 1993, 2006). It has since been applied to various topics in mathematics, including whole-number multiplication (Izsák, 2005), functions (e.g., Izsák, 2004; Moschkovich, 1998), and integrals (Jones, 2013). From this perspective reasoning is supported by diverse, fine-grained knowledge resources and more novice knowledge evolves into more expert knowledge through processes such as the construction of knowledge resources that are sensitive to context for activation, refinement of contexts in which resources are applied, and reorganization that can involve forming new connections among some recourses and loosing connections among others. In the present study, we were particularly interested in knowledge resources students evidenced when reasoning about multiplication in the context of definite integrals and across both *I-O-M* and *P-O-M* situations.

Methods

We conducted three one-to-one, semi-structured, think-aloud interviews to assess how five students enrolled in the first-semester calculus-based physics course at the selective university reasoned about the integral concept in different problem situations, with particular attention to whether and how they constructed the product layer. Each interview lasted approximately 1 hour.

The *first* interview examined physics students' reasoning about multiplication in I-O-M and P-O-M problem situations. This interview consisted of three situations. None of the tasks mentioned integrals, but for each situation one could first multiply and then add to construct a total. We paid close attention to how students assigned units of measurement to the quantities and how they used the later to identify unknown quantities in each problem situation. The *second* interview examined physics students' reasoning when connecting definite integrals to I-O-M and P-O-M problem situations. In addition, we examined how students identified multiplicative structures and related them to graphical representations. The *third* interview intended to examine how students reasoned about the definite integral concept in I-O-M and P-O-M situations, but we were only able to pursue the I-O-M situations in the allotted time. Again, we paid close attention to how students reasoned about multiplicative structures and graphical representations as a part of the definite integral concept.

We recorded the interviews using two cameras, one to capture the student (body movements, hand gestures, etc.) and the interviewer and one to capture the student's written work. We collected all written work at the end of each interview. We used a computer to synchronize the two video recordings into a single file and transcribed the interviews verbatim.

We watched videos for all five students side-by-side with the transcripts, checking the transcripts for accuracy. All students had trouble at one point or another coordinating multiplication with the "product layer" and definite integrals. After multiple viewings of the interview data, it became clear that Iliana's data was particularly useful for our research questions because she first struggled to reason about multiplication in the context of the integration tasks, but then constructed the product layer appropriately. Her case is all the more noteworthy because she self-reported strong performance in calculus (Calculus, A+; AP Calculus AB, 5; AP Calculus BC, 5). We generated summaries of the Iliana's reasoning that attended to her spoken language, hand gestures, and drawings as she worked on each task.

Results

Iliana's Work

Due to space limits, we focus on Interview 3. During Interview 1, Iliana relied on recalled formulas for transforming units when determining what to multiply — for instance, she recalled that $ft/sec \cdot \text{sec} = ft$ and that $cm^2 \cdot cm = cm^3$. The main additional result from Interview 2 was that, in the context of a problem that provided equally-spaced, cross-sectional areas of a liver and asked for an approximate volume, Iliana multiplied cross-sectional areas by lengths to construct small pieces of volume which she then added, appropriately. At the same time, when using integral notation to express her reasoning, she included both Δx to denote small lengths and dx (Figure 2). When asked to interpret dx, she explained:

Iliana: I think maybe that just like everything's like in respect to x, I don't know, honestly, like that's always like added at the end and it's like I, we've definitely like gone over exactly why.

Figure 2. Iliana's definite integral representation (Interview 2)

Thus, in the Liver task Iliana perceived products in terms consistent with normatively correct Riemann sums and integrals but, for her, the dx notation did not indicate a factor to multiply. Interview 3 would reveal additional challenges experienced by Iliana.

Data. Task 1 from Interview 3 asked Iliana to approximate the total force from water pressure on a tank wall. The task included a picture of the 3ft-by-4ft wall and stated that the pressure on the wall varied with the depth, x, according to $P(x) = 15x$. A normatively correct integral would be $\int_0^3 4 \bullet 15x dx$. The physics course had yet to cover the relationship between pressure and force, and we anticipated that the context would be novel for Iliana. Indeed, she said she had to think about how pressure relates to force. She then integrated $15x$ "on the bounds from zero to x, x being whatever the depth is" to calculate the "sum of all the surface areas." Finally, she explained that she thought the pressure pointed down to the bottom of the tank (Figure 3). For this reason, she did not "feel the 4 feet of the width has anything to do with that."

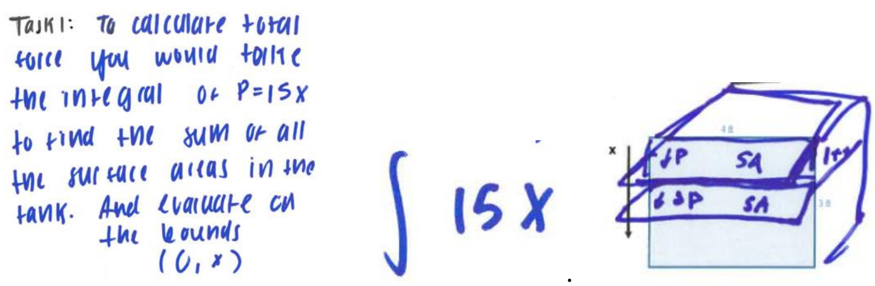

Figure 3. Iliana's work on calculating the total force (Task 1)

When the interviewer asked about force acting in another direction, Iliana continued to view pressure vertically. At the same time, she was not confident and reported that she did not know if force and pressure were related through a derivative/antiderivative, where the width of the wall would come in, and how to interpret the phrase "pressure across a surface area." Finally, Iliana sketched an appropriate graph for $P(x) = 15x$ but did not discuss area under the curve.

Analysis. Iliana's work on Task 1 provided access to how she approached integration in a novel situation, not in one where she recalled certain multiplicative relationships or formulae. An interconnected set of ideas (i.e., uncertainty about the pressure and force relationship, downward pressure) impeded her construction of products used in normatively correct Riemann sums and definite integrals. First, although the problem statement asked explicitly about pressure on the wall, Iliana consistently focused on downward pressure. This orientation side-stepped the need to consider small pieces of wall area, $4dx$, as one factor in a product. Second, similar to her work in Task 2 summarized above, Iliana did not view multiplication by dx as transforming units. Rather, she integrated the pressure function, $P(x) = 15x$, to get a total pressure. Third, similar to her work on the Liver task discussed above, she did not treat dx as a conceptually essential piece of the definite integral. In fact, she omitted the notation. Thus, how she oriented pressure in the situation directed her attention away from multiplying small areas of the wall ($4dx$) with pressures ($15x$), and it was unclear whether she noticed the absence of unit transformation through multiplication. If transformation of units through multiplication were central to her understanding of definite integrals, integrating a pressure function to get total pressure might have indicated to her that something was off.

Data. Task 2 asked Iliana to consider an automobile accelerating at $a(t)$ (m/sec^2) for 9 seconds and to find the velocity after 9 seconds. First, Iliana recalled an appropriate velocity formula, $v = v_0 + at$. She substituted 0 for v_0 and 9 for both a and t, arriving at 81 m/s. She then wrote $v = 9a$ (Figure 4). When asked how she knew to multiply, Iliana responded:

Iliana: a is the acceleration, which is just equal to, over here, like the amount that the, the amount that the, the like velocity is changing like per second, which in terms of units is meters per second second. Because meters per second is the unit for velocity, and then you add an extra second in the denominator because it's the change of velocity per second. So then when you multiply by time, which is just seconds, it cancels out and you have this, the unit for velocity again.

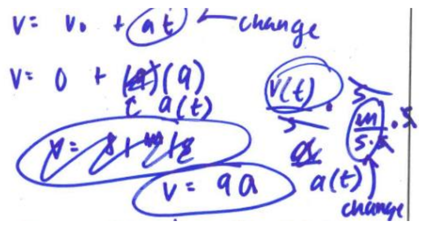

Figure 4. Iliana's work on calculating the velocity value for 9 seconds (Task 2)

Finally, Iliana mentioned the derivative/antiderivative relation between the acceleration and velocity quantities: "velocity is the antiderivative of acceleration", that is, "given like acceleration, let's say, $8x$, you can think that integral of acceleration evaluated on the time period zero to t, where t is the number of seconds that the system did."

Analysis. Iliana's work, including her discussion of anti-derivatives, suggested prior experience with problems relating velocities and acceleration. She introduced multiplication through a recalled formula and formal cancellation of units. From her discussion, it appeared that she recalled appropriate units for acceleration and for velocity and then reasoned about how to get from one to the other by cancelling units of time. Given that at one point she substituted 9 for both a and t, it is likely that she presumed a to be constant. This could explain why she did not use multiplication to construct small changes in velocities which could then be added, in contrast to the way she did use multiplication to construct small pieces of liver volume.

Data. Task 3 asked Iliana to explain the meaning of the definite integral and its interpretation in Task 1 and 2. Iliana stated that "it's harder than I thought to quantify what [the integral] exactly means." She then explained that the integral means "the sum of like a function per like change" and that "d just means like the change." (Figure 5). She added that "from a to b" means the summation of "those like areas of $f(x)$ per each change, like d of x." She concluded by saying "you are thinking of each small area and then adding them up." An exchange later she added, "Like change for each x, like per x, that's the change."

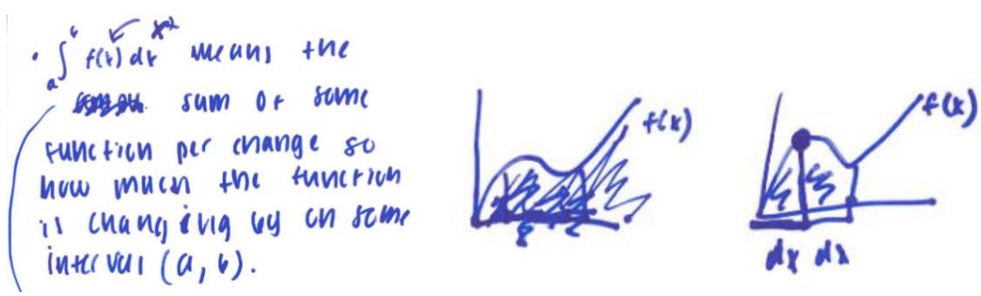

Figure 5. Iliana's definition of the definite integral and graphical representation of $f(x) \cdot dx$ (Task 3)

The interviewer then asked Iliana to explain her understanding of areas. Iliana graphed a function, shaded the area below it, and said "the summation kind of means like, you want to find what's under here, right?" When interviewer then asked Iliana where she saw dx and the product $f(x)dx$, Iliana drew horizontal brackets to indicate instances of dx and a large point on her graph to indicate $f(x)dx$ (Figure 5).

Analysis. Iliana demonstrated several ideas relevant to explaining definite integrals. These included summation of pieces, attending to changes in x, and areas. At the same time, there was no evidence that products were central for her. Similar to her work on the Water Pressure task, she talked about summing the function $f(x)$, not $f(x) \bullet dx$, and her discussion of change for each x and change per x suggested that she thought more in terms of a correspondence between values of $f(x)$ and instances of dx than a product of the two. Her plots of dx and $f(x)dx$ provided further evidence that she did not connect this notation to products and small areas.

Data. Next, Iliana interpreted the definite integral in the context of the acceleration task. First, she wrote appropriate integral notation and stated: "It's actually really the same thing, a of t per change in time. And then you're summing for, to evaluate the change of velocity per the amount of time done basically." Then she produced a graph corresponding to her explanation and drew the large point to indicate where she saw the expression $a(t)dt$ (Figure 6, middle).

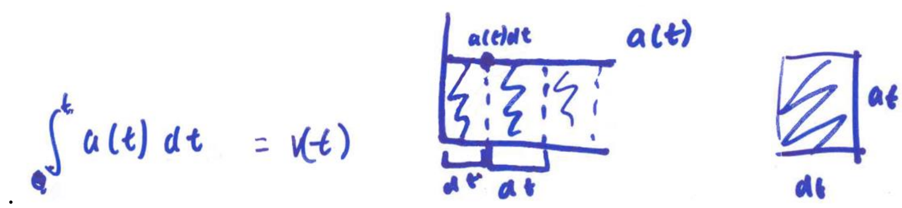

Figure 6. Iliana's meaning of the definite integral (Task 3)

Right after this answer, Iliana expressed some confusion about the dt term in her expression:

Iliana: Now, I'm confused what dt exactly is like representing. Because like sometimes it just feels like it's just there and doesn't really represent something. But like, obviously I know it does…. it's just like something that's kind of there, I don't know.

When the interviewer asked "What are you indicating when you add the squiggly lines?", Iliana responded the "area under the line" which was the "change in velocity for that time period."

Then the interviewer asked the following question, which made multiplication explicit:

Interviewer: Does this area that you indicated here (points to the squiggly lines, Figure 6, middle), and that expression $a(t)$ multiplied by dt has any relation or no for you?

After thinking for few seconds, Iliana changed her reasoning:

Iliana: Well, that is the area. Okay. Maybe this is making a little more sense now actually. Dt multiplied by a(t) gives you this area because it's just a box and you know that like to find an area of a rectangle is just like the base times height or whatever, you know. So, when you multiply those together, that gives you an area and the summation is just saying the sum of each of those areas for one, some time period. So, this isn't a point (points to $a(t)dt$). This is all of this, (points to shaded region) the a of t d of t is the squiggly lines.

Iliana shifted her reasoning and produced a new rectangular area picture (Figure 6, right), where she indicated $a(t)$ to be the height of the rectangle and dt its width. She reported that connecting $a(t)dt$ to an area was new to her. Finally, Iliana stated that integration will give her the velocity quantity and pointed to the sum of areas under the curve as the total velocity value on that time interval determined by the bounds in the integral notation. There was not time to return to the Water Pressure problem.

Analysis. Initially, Iliana interpreted the integral of $a(t)dt$ in ways similar to her interpretation of $f(x)dx$. She thought of a summation of pieces that corresponded to instances of dt but that did not think of those pieces as products for which dt was a factor. Just as she interpreted $f(x)dx$ as point, so again did she interpret $a(t)dt$ as a point. In both cases, associating dx and dt with intervals on the horizontal axis was insufficient to cue multiplication for Iliana. Once the interviewer introduced interpreting $a(t)dt$ as a product, Iliana generated a normatively correct interpretation that coordinated multiplication with the definite integral for the first time during the interviews.

Remaining Four Students. The remaining four students also demonstrated a range of resources for reasoning about multiplication—including formal unit cancelation manipulations—and evidenced a range of challenges reasoning about multiplication in the context of definite integrals. For the Water Pressure task, a second student integrated $15x$, one student considered integrating P/A, one student considered the product $15x \bullet 4x$, and one student avoided integrals altogether and reasoned instead about average pressure. We also observed further instances in which students did not associate dx notation with small horizontal lengths.

Discussion and Conclusion

The present study extends prior reports (e.g., Sealy, 2014) that students have trouble with the "product layer" when constructing definite integrals. In particular, we considered the possibility that students' difficulties might reflect where and how they use knowledge resources related to multiplication, as well as how they understand integral notation. Data on Iliana provided particularly good access to knowledge resources used by one high-achieving student. We found that Iliana had a range of resources for modeling problem situations with multiplication—including recalled formulas and strategies for transforming units—as well as a range of understandings about integrals—including thinking of areas under curves and derivative/anti-derivate relationships. We are not claiming that this is an exhaustive list of Iliana's resources related to multiplication and integration, only that they were ones central to her progress by the end of Interview 3. Iliana's work on the Liver problem demonstrated a case where she drew on prior knowledge about multiplication to form small pieces of volume that she then added. Thus, in some contexts she reasoned in ways consistent with Riemann sums but was not able to connect this reasoning to normatively correct integral notation. A main issue was her uncertainty about the meaning of the dx notation. Iliana's work on the Water Pressure problem demonstrated a case where the way she perceived novel quantities (i.e., downward pressure) directed her attention away from products. Her challenge was compounded by the fact that she did not perceive a product in the integrand that transformed units, in this case units of pressure. Finally, at the end of Interview 3, and in response to some direct suggestions by the interviewer, Iliana perceived dt for the first time as a factor in a product and combined thinking about products, transformation of acceleration into bits of velocity, and integral notation appropriately. Such complex coordination provides new evidence for why forming the "product layer" can be challenging for students. Our results are based on a small sample, but they do suggest that research on integration should pay more explicit attention to the knowledge resources students have for modeling situations with multiplication and how those combine with understandings of the definite integral.

Acknowledgments

This research study was supported by the Graduate Student Research Competition Funding at Tufts University, Graduate School of Arts and Sciences.

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