Invoking Conceptual Change in Adult Learners: "Seeing" Fractions Differently

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*I was charged with developing and implementing a course redesign to improve low success rates in the lowest university-level developmental mathematics course (median ACT score = 17). As part of the initial implementation of the course redesign a diagnostic assessment was given to understand the nature of students' conceptions of foundational concepts. Student responses on several items indicated that over 25% of students had a dominant part-whole conception of fractions; not seeing fractions in terms of measures. This dominant part-whole conception of fractions also influenced how students reasoned about items involving other concepts (e.g., addition of fractions and ratio comparisons). In this preliminary report the results that led to our conclusions, the instructional intervention we used to move students' conceptions of fractions forward, and the impact of the intervention, based on the results of the post-assessment, on students' conceptions of fractions will be shared.*

*Keywords:* developmental mathematics, conceptual change, fraction conceptions

## Introduction

At the university in which I teach the five-year success rates in the lowest developmental mathematics (DM) course has consistently hovered between 50% and 60%. Despite these low success rates, not much has been known about the nature of these students' conceptions of foundational concepts besides their over reliance on procedures without knowing the mathematical "why" (Stigler et al., 2010). Charged with improving success rates in this course it was important to first understand the "roots" of their struggle with mathematics. A thirteen-item multiple-choice diagnostic assessment (n = 230) was given to assess students' current conception of place-value, fractions, area/perimeter, and proportional relationships. While the results suggested that students' faced challenges within each of these constructs the most striking and revealing were those related to their understanding of fraction.

Existing research with students at various levels of academic experience suggested that students' knowledge of fraction magnitude, or as a number, is highly associated with their computational skills (Schneider & Siegler, 2010), algebraic knowledge (Booth et al., 2014), and general mathematics achievement (Torbeyns et al., 2015). Seeing fractions as a magnitude requires students to engage in multiple levels of unit coordination that involve relationships between quantities and various size of units. We had previously conducted research with $4^{th}$ through $7^{th}$ graders and were curious as to whether DM students' conception of fractions was similar. We had previously found that elementary and middle school students often struggled with fraction magnitude concepts because their dominant conception of fractions was as a part-whole relationship between two quantities without regard for the size of the unit represented by the denominator of the fraction. An example of student reasoning reflecting this part-whole relationship is shown in figure 1. When asked to compare the fractions 5/6 and 7/8 the student partitioned the whole unit into a quantity of equally sized pieces represented by the denominator of the fraction (i.e., 6 and 8) and shaded the quantity of pieces represented by the numerator (i.e., 5 and 7). A student with a dominant part-whole conception of fractions will reason that the two fractions are equal because the quantity of missing pieces is the same (i.e., 1); not considering the size of the unit attached to each of those quantities, sixths and eighths, respectively. Students who do reason about the size of the units demonstrate elements of fraction-as-measure conception (Wilkins & Norton, 2018). We were curious as to whether DM students' conception of fractions was similar. We hypothesized that if DM students had a dominant part-whole conception of fractions it may also influence how these same students reason about operations involving fractions and proportional relationships.

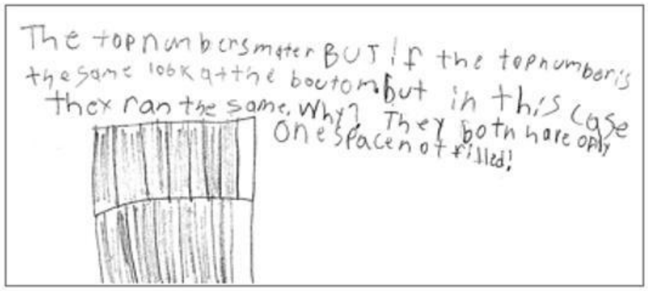


*Figure 1. Student reasoning to compare 5/6 and 7/8.*

To explore the nature of DM students' current conceptions a thirteen-question multiple-choice diagnostic assessment was given at the beginning of the DM course  The results of three items from the diagnostic is shown in Table 1. Q4 is an item that was used to determine whether students had a dominant part-whole conception of fractions.  Q9 and Q12 are also shared because it was hypothesized that  student with a dominant part-whole conception of fractions would reason about these two tasks in a particular manner. The notation (C) in Table 1 represents the correct answer while the bolded response indicates the response in which a student with a dominant part-whole conception of fractions would most likely choose. It is important to note that given that Q4 was a multiple-choice we do not know whether the students who chose the correct answer had a meaningful conception of fractions or simply relied upon a whole-number relationship (e.g.,7 > 5). However, we believe we can infer that if a student chose "they ate the same amount" they were at best utilizing a part-whole relationship.

*Table 1. Sample diagnostic questions  (n = 230)*

| # | Questions | | | |
|---|---|---|---|---|
| Q4 | Two pizzas are the same size. Carlos ate 5/6 of one of the pizzas and Terrell ate 7/8 of the other pizza. Who ate more? | | | |
| | Carlos | Terrell (C) | **They ate same amount** | Impossible to know |
| | 30.43% | 41.30% | **26.52%** | 1.74% |
| Q9 | Thomas ate 3/4 of a whole medium pizza and Lydia ate 5/8 of a whole medium pizza. Together they ate how much of a whole medium pizza? | | | |
| | **8/12** | 11/8 (C) | 8/8  (C) | Cannot determine |
| | **40%** | 45.65% | 9.13% | 5.22% |
| Q12 | Mix A: 3 cups of OJ to 4 cups of water     Mix B: 6 cups of OJ to 8 cups of water Which mixture (A or B) will be juicier? | | | |
| | **Mix A** | Mix B | They will be the same (C) | Impossible to determine |
| | **34.44%** | 14.50% | 49.85% | 1.21% |

As part of our preliminary analysis of the diagnostic results a chi-square test for independence was conducted to examine our hypothesis that students answering Q4 in a way suggesting a dominant part-whole conception (i.e., two fractions are equal) would also answer Q9 (i.e., 8/12) and Q12 (i.e., Mix A) in a predictable manner. The results of this brief analysis confirmed our initial hypothesis.

*Table 2. Chi-square analysis. $X^2$ (230,1) * $p<.05$*

| | Q9 | Q12 |
|---|---|---|
| Q4 | 6.6868 | 16.4698 |
| | $p = 0.0097$* | $p = 0$* |

**Interventions**

As the preliminary analysis of the diagnostic assessment revealed many of the DM students relied on a part-whole conception of fractions to reason about fraction comparisons. The first challenge of the course redesign was to develop learning experiences that confronted the limitations of a part-whole conception while also moving them towards fraction-as-measure conceptions. Guided by two frameworks, Conceptual Change (Vosniadou, 2013) and Progression of Fraction Schemes (Wilkins & Norton, 2018) a set of instructional activities were designed. One of the elements of the conceptual change framework is that "knowledge acquisition is not always a process of enriching conceptual structures. Sometimes the acquisition of new information requires the radical reorganization of what is already known" (Stafylidou & Vosniadou, 2004, p. 504).

A radical reorganization of DM students' fraction conceptions required explicit attention to the meaning and role of the unit. This was addressed in two ways. First, tasks were designed to engage students in mental activities that supported the development of fraction-as-measure conceptions (Wilkins & Norton, 2018). These tasks engaged students in the mental activities of partitioning, iterating, and disembedding. For example, as part of one the tasks we asked students to fold (partitioning) fraction strips representing the whole unit into equally-sized pieces such that only the unit fraction was showing. Then we asked them to iterate the unit fraction to determine the length of copies of the unit fraction. For example, as shown in figure 2, consider a paper strip representing 1 whole unit of length.

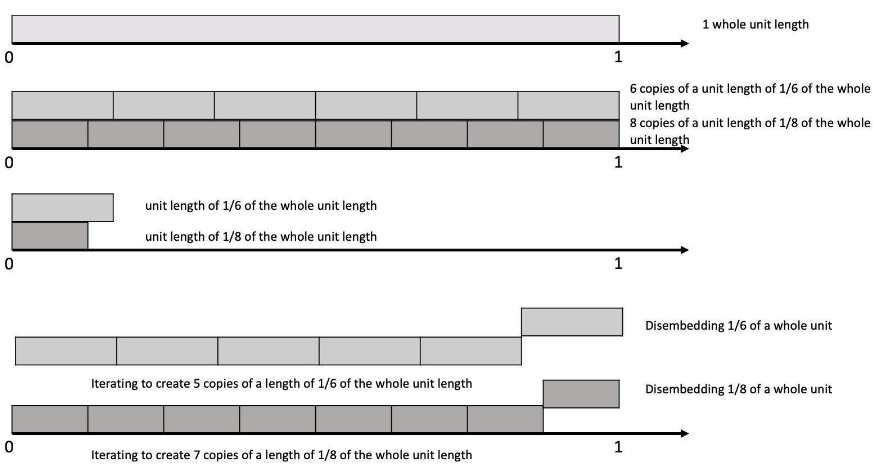


*Figure 2. Coordination of units to engage in a measurement scheme for proper fractions.*

The first action involves *partitioning* the whole into equally sized lengths and recognizing that each of those lengths represents a unit fraction (e.g., 1/8). *Iterating* the unit fraction, a given number of times represents a length that is multiple copies of the length of the unit fraction. For example, a length of 7/8 of a whole unit is 7 copies of the length of the unit fraction 1/8.The *disembedding* action happened to determine the "missing part" from the whole. For example, as shown in figure 2, the missing part of 7/8 of a whole unit is a unit fraction of 1/8 because 8 copies of the unit fraction 1/8 is the same length as the whole unit.

The second way in which the unit was made explicit is that fractions were represented using a novel notation, numeral-unit-name notation. For example, the fraction 7/8 was written as 7 eighths using the numeral to represent the quantity and the unit-name to represent the unit. This decision supported a learning acquisition goal and a connection goal. Students were introduced to the idea that the morpheme "ths" indicated that the whole unit is partitioned into a quantity of equally-sized lengths represented by the word preceding the morpheme (e.g., eighths). It also supported efforts to utilize overarching principles to connect operations across numerical and algebraic expressions. For example, instead of seeing an addition problem such as 3/4 and 5/8 in terms of "getting a common denominator" our goal was for students to see adding fractions as combining quantities of the same size of unit. Thus, 3 fourths and 5 eighths cannot be combined because the size of the units are not the same. However, utilizing an equal exchange of 6 eighths for 3 fourths the numbers 6 eighths and 5 eighths can be combined because the size of the units are the same (6 eighths + 5 eighths = 11 eighths).

While not a significant part of this preliminary report, it is important to note that making the units explicit was also an important aspect of the development of understandings related to equivalent ratios and proportional relationships in the DM class. For example, consider the ratios represented in Q12 of the diagnostic test. The ratio 6 cups of OJ to 8 cups of water is the same juiciness as 3 cups of OJ to 4 cups of water because it represents 2 copies of the ratio 3 cups of OJ to 4 cups of water [2 x (3:4) = (6:8)]. These multiplicative relationships between ratios were made explicit using ratio tables and double number lines.

## Results

After one iteration of the course redesign the success rate in the DM course increased from 56% to 80%. Despite a significant shift of the DM content to more arithmetic concepts, as opposed to algebraic concepts, the failure rate in the ensuing general education mathematics course also decreased slightly, 31.40% to 28.74%.

A post-assessment involving the same questions as the diagnostic assessment was given ten weeks after the fraction concept module. Students were not given advanced notice that they would be re-taking the diagnostic assessment. The percentage of students that correctly answered each question on both the pre-assessment and same post-assessment was computed. Only the results of students ($n = 117$) who completed both the pre- and post-assessment are included to remove potential variation in percentage correct resulting from differences in pre- and post-assessment participants. A difference of proportions *z*-test was conducted to measure whether the change in percentage correct from pre- to post-assessment was statistically significant. Level of significance was set at $p < .05$. Results of this analysis are shown in Table 3.

*Table 3. Comparing pre- and post-test results.*

| # | Question | % correct (pre) | % correct (post) | p |
|---|---|---|---|---|
| Q4 | Two pizzas are the same size. Carlos ate 5/6 of one of the pizzas and Terrell ate 7/8 of the other pizza. Who ate more pizza? | 47.86 | 61.54 | .035* |
| Q9 | Thomas ate 3/4 of a whole medium pizza and Lydia ate 5/8 of a whole medium pizza. Together they ate how much of a whole medium pizza. | 52.14 | 76.92 | .000* |

The results suggest that there was growth in students' fraction-as-measure conceptions. We acknowledge, as discussed earlier, that comparing the fractions 5/6 and 7/8 requires a high degree of unit coordination sophistication (i.e., disembedding 1/8 and 1/6 to describe the measure of the "missing piece") and reasoning. We were specifically curious as to the change in percentage of students who indicated that the two fractions were equal. This percentage shifted from 26.5% on the pre-assessment to 13.68% on the post-assessment. We also conducted the same chi test of independence on the results of the post-assessment as we did on the pre-assessment and found the same relation between those who answered Q2 as equal and Q9 as 8/12, $X^2(1,117) = 7.87$, $p = .005$*.

Three other questions showed a statistically significant relation between indicating that 5/6 and 7/8 were equal and an incorrect response. These questions included Q8 (identifying measure on a tape measure), $X^2(1,117) = 24.159$, p = .0012* , Q11 (proportional reasoning involving distance traveled and a fraction quantity), $X^2(1,117) = 7.57$, $p = .006$*, and Q13 (proportional reasoning requiring unitizing), $X^2(1,117) = 5.258$, $p = .022$*.

## Discussion

Consistent with what others have found (Pesek & Kirshner, 2000) as well as elements of Vosniadou's (2003) Conceptual Change Framework, it is challenging to move conceptions forward once students have procedures and conceptions already in place that interfere with the intended concept advancements. Even after the intervention 13.68% of DM students still indicated that the fractions 5/6 and 7/8 were equal and many still used this part-whole fraction relationship to add fractions.

This study gave us an opportunity to reflect on our teaching practices. We found that we need to spend more time engaging students in iterating unit fractions. From the analysis of the pre-and post-assessment we concluded that we need to begin these actions with whole numbers while also making more explicit the multiplicative relationship that exists between quantities and size of units of decimal numbers. Inherently numbers have an implied multiplicative relationship. For example, 0.5, is 5 x (1/10). Same is true with fractions, 4/8 or 4 eighth is also 4 x (1/8). Writing out the multiplicative relationship sheds light on the meaning of the unit fraction in relation to the whole unit and that 4/8 is 4 times greater than 1/8.

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