What is the Correct Amount of Change? A Case Study on Kala's Covariational Reasoning

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Students' understanding of the mathematics of change has been a crucial topic in the teaching and learning of precalculus. Various studies have indicated that covariational reasoning – reasoning about two quantities changing together—is critical for constructing productive meanings for rate of change. The Bottle Problem is one way teachers have supported students' covariational reasoning. We report on a case study of one student – Kala – an Applied Precalculus student. Through analysis of her work on the Precalculus Content Assessment (PCA), course assignments, and a cognitive interview, we report on her reasoning with amounts of change—a level of covariational reasoning. As a result, we highlight the importance of using appropriate amounts of change to analyze situations covariationally. This study contributes to literature on students' development of productive amounts of change reasoning.

Keywords: Covariational Reasoning, Precalculus, Cognition, Bottle Problem

Supporting students in constructing productive meanings for rates of change has been an aim for researchers for decades (Thomspon & Carlson, 2017). From middle school on, various materials aim to support students in reasoning with rates (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ellis et al., 2015; Johnson, 2015; Tasova, 2021; Yu, 2022). The Bottle Problem (e.g., Carlson et al, 2002) is one well known example. Nevertheless, students still enter their post-secondary courses struggling with ideas of rates of change (Carlson, Oehrtman, & Engelke, 2010). The Precalculus Content Assessment (PCA) is one assessment that not only helps identify students' precalculus content knowledge, but also attends specifically to students' covariational reasoning—reasoning about two quantities changing in tandem (Carlson et al., 2002) (heretofore CR). Our research topic was to explore students' ways of reasoning when they attempt to reason about changing quantities quantitatively but struggle to construct or interpret representations. This interest led us to explore Kala's reasoning via her and her classmates' results on the PCA and the Bottle Problem. Our results provide insights into the importance of selecting appropriate *amounts of change*—one of the levels of covariational reasoning. We conclude by discussing the implications of her reasoning to the research on CR.

Background and Theoretical Framework

Covariational Reasoning and the Bottle Problem

Carlson et al. (2002) illustrated their framework for covariational reasoning with the Bottle Problem, a problem used and adapted in several settings with various populations. In Carlson et al.'s (2002) version of the Bottle Problem, students receive an image of a cross-section of a spherical bottle with a cap and asked to imagine the bottle filling with water and to "sketch a graph of the height as a function of the amount of water in the bottle." Their resulting framework consisted of five mental actions: *coordination of change*, *directional change*, *amounts of change*, *average rate of change*, and *instantaneous rate of change*. For example, students who coordinated the quantities might say that the height is changing as the amount of water is changing. Students with *directional reasoning* might say that the height is increasing as the amount of water is increasing. Students with *amounts of change reasoning* might say that for equal changes in the amount of water added, the water is increasing by increasing amounts. From these descriptions, we can recognize increasing precision about how the quantities change.

In this study, we were interested in exploring more in depth about how students develop CR. Johnson, Castillo-Garsow, Moore and colleagues (e.g., Castillo-Garsow, Moore, & Johnson, 2013; Johnson, 2015; Moore et al, 2019) have already done work in this area, reporting how students visualize quantities changing (such as the speed of the change or considering discrete vs. continuous change) and how their meanings for graphs can impact their reasoning. Another relevant result is from Stevens (2023), in which students compared the steepness of lines on a graph (i.e., the slope) to determine increasing rates of change.

Precalculus Content Assessment (PCA)

The goal of the Precalculus Content Assessment (PCA) is to provide a means to evaluate how effective a curriculum and its instruction are in setting students up to be successful in calculus (e.g., effectiveness of College Algebra (Carlson, 2010)) and link between AP Calculus and PCA (Meylani, 2011). In Carlson's (2010) study of 902 precalculus students, the average score on the PCA was 41% with a reliability coefficient of 0.73. The average score on the PCA for the covariation reasoning abilities was 50% with a reliability coefficient of 0.46. Though these reports pull out students' covariational reasoning problems (which we also do in this study), it is important to note that a factor analysis on the PCA indicated the use of a single total score on the PCA may be the "most psychometrically defensible method of scoring the instrument" (Jones, 2021, p. 78).

Intellectual Need

Because this study is attempting to explore in depth a student's CR, Harel's (2008) *intellectual need* became a key idea in the analysis of the data in identifying reasons a student might engage in covariational reasoning. Harel (2008) defined *intellectual need* as the need to reach an equilibrium (i.e., no state of perturbation) by learning a new piece of knowledge. For instance, when choosing to graph a straight line or a curve, a student may recognize a need to attend to the rate at which quantities are changing in the situation to reach a state of equilibrium.

Methods

Subjects and Setting

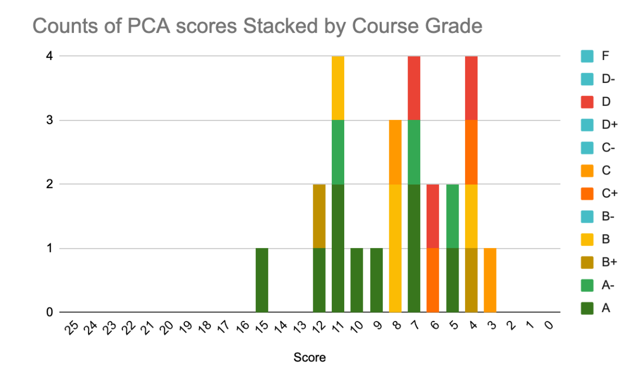
This study was conducted with an Applied Precalculus class of 39 students at a medium-sized university in the northeastern US. The coordinated course had students with various majors (but not Mathematics or Engineering). The students were assigned four different assignments throughout the semester, one of which included an extended version of the Bottle Problem from Moore's *Advancing Reasoning* NSF project (see results for details) and a prompt to submit a video explaining their reasoning. 25 of the students completed the optional PCA in class at the end of the semester. One student, Kala, volunteered to be part of a semi-structured clinical interview (Clement, 2000) after the semester ended in which she, once again and with no advance knowledge, went through a subset of the problems from the four classroom assignments.

Analysis

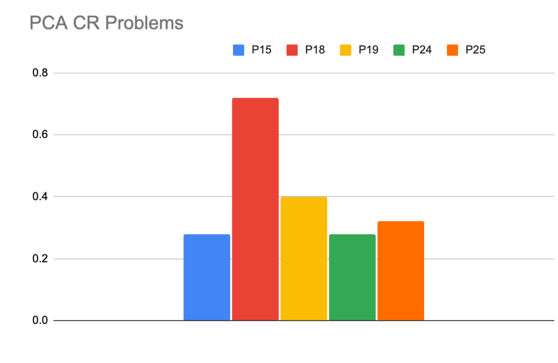
We recorded PCA responses and scanned student work, we downloaded the classroom assignments from the course's LMS where students had uploaded pdfs and video files, and we video-recorded and transcribed the interview (camera and screen recording of iPad). We analyzed the PCA results by question (correctness, summary data, distribution of responses), and then focused on the problems Carlson, Oehrtman, & Engelke (2010) identified as relevant to CR (heretofore PCA CR problems). We used Carlson et al.'s (2002) CR framework to analyze the student work on those problems, and we then compared student's work with Kala's work. We then analyzed Kala's student work and video associated with the Bottle Problem assignment, coding her *amounts of change reasoning*, which we analyzed further using Kala's interview on the Bottle Problem.

Results

Kala's Overall PCA Results Compared with Classmates



(a)



(b)

	Total PCA by problem	PCA CR by problem	Total PCA by student	PCA CR by student
max.	76%	72%	60%	100%
min.	0%	28%	12%	0%
average	31.2%	40%	31.2%	42.8%
mode	32%	28%	44%	40%
median	32%	32%	28%	40%
var.	0.0496	0.0344	0.016	0.0610
st.dev.	0.2227	0.1854	0.1264	0.2470

(c)

Problem	Non-CR categorizations
15	rate of change, general non-linear, graphical, contextual
18	algebraic, contextual
19	rate of change, general non-linear, graphical
24	general non-linear, graphical
25	rational, algebraic

(d)

Figure 1: (a) Total PCA scores with course grade (b) % correct for PCA CR problems (c) Summary statistics by problem/student for the total PCA and for PCA CR problems and (d) non-CR categories for PCA CR problems.

The PCA results from the 25 students' scores are summarized in Figure 1a. Kala was one of four students who scored 4 points (out of 25) and received a B+ in the course. Figure 1b shows the results of the student responses on the PCA CR problems. Kala was one of the 7 students (i.e., 28% of the class) who got Problem 15 (P15) correct on the PCA, but all her other PCA CR problem responses were incorrect. Problem 15 was the most like the Bottle Problem, because it involved choosing a geometric 3D shape that could represent a given height vs. volume graph (other PCA CR classifications in Figure 1d). Figure 1c provides PCA summary statistics. In the first two columns, we see 76% of students answered the most correctly answered PCA problem correctly, while it was 72% for the max PCA CR problem. Also, students performed better on average on the PCA CR problems than the total PCA assessment (31.2% vs 42.8%). Kala scored a 16% on the PCA and a 20% on the PCA CR problems alone, both below average scores.

Kala's PCA CR Problems Results Compared with Classmates

Kala only got one of the five PCA CR problems correct (P15). In this section, I (i) state how she answered the problems (ii) compare her responses to her peers' responses, and (iii) report on whether (and how) the content was covered in the Applied Precalculus course she took.

On P15, students chose a shape that represented a given volume-height graph. Most students chose shapes that had the opposite rate of change as what was represented in a given volume-height graph. Kala (no written work) was part of the 28% of students who answered correctly. In class, students did a very similar task (i.e., the Bottle Problem).

On P18, students needed to determine the *direction of change* and an extreme function value from given formula. P18 was the second most correctly answered PCA problem. Most students substituted values to solve this problem. Kala (no written work) got P18 incorrect. For P19, 40% of students correctly described the behavior of a function when given a graph. Only one student (not Kala) incorrectly identified the *directional change*, but the most common response included the opposite rate of change. Kala and one other student identified a constant rate. In class, students determined *directional change*.

For P24 and P25, students described the end behavior of a function given a graph and a rational function definition, respectively. 28% and 32% of students correctly identified end behavior, respectively. Kala did not correctly identify any end behavior statements for either problem. End behavior was not explicitly taught in the course.

From Kala's work on these problems, we identify the following characteristics. Kala did not demonstrate a way to reason about the *direction of change* when given a function definition (P18), but she could when given a graph (P19). Kala did not demonstrate a way to reason about the end behavior of a function when given a graph or a function definition (P24/P25). Kala inconsistently reasoned about rates of change when given a graph (P15/P19). To gain more insight into Kala's reasoning, we turn to her work on Assignment 3 and her cognitive interview.

Assignment 3 (The Bottle Problem) Results

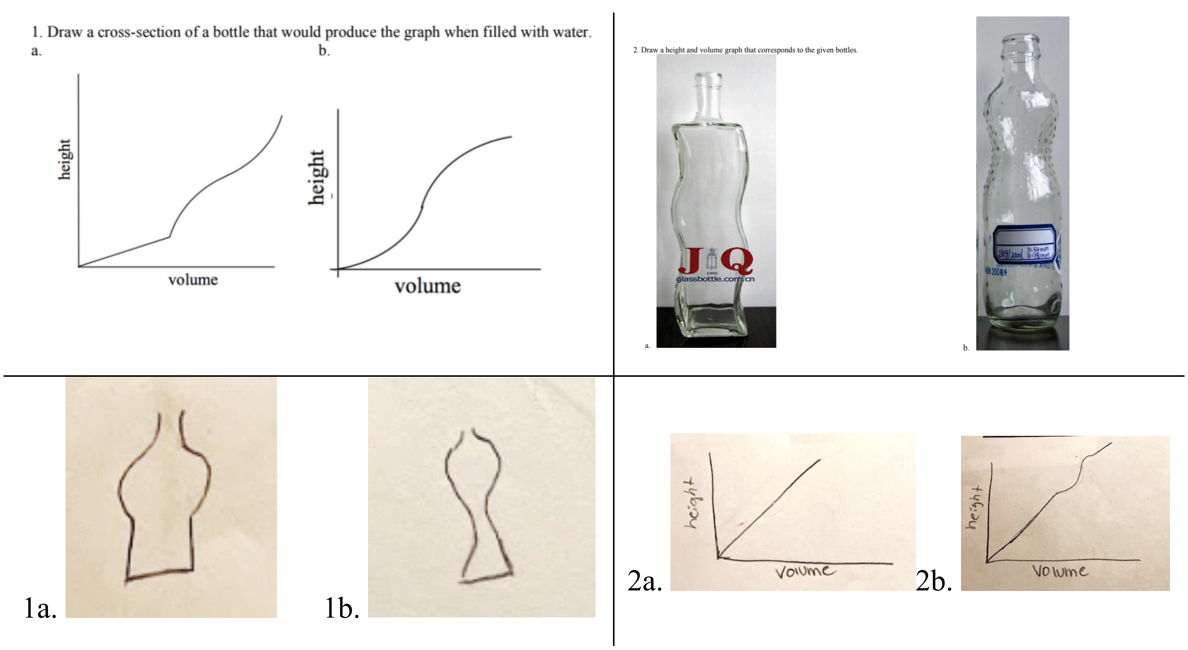
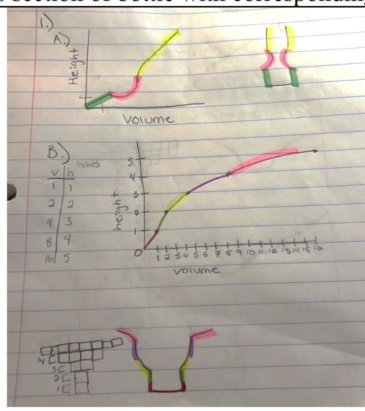


Figure 2: (top) Assignment 3: Part 1 and (bottom) Kala's solutions.

1a. Height-Volume graph with corresponding cross section of bottle.
1b. Height-Volume graph; the water doubles in volume for each additional inch of height.
1c. Height-Volume graph; for each inch in height the bottle increases, the volume of the bottle increases by two more inches cubed in volume that the previous increase.
2. Cross section of bottle with corresponding Height-Volume graph.



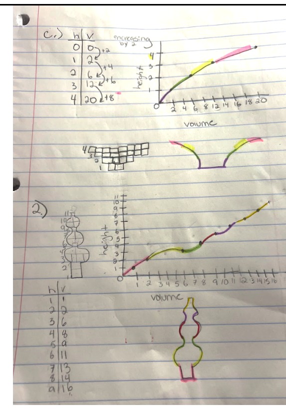


Figure 3: Kala's Assignment 3: Part 2 solutions with summary of problem statements in top row.

During the semester, students worked on a modified version of the Bottle Problem. Part 1 (Figure 2) is above with Kala's solutions. For 1a, her cross section of the bottle is a normative solution. For 1b, there is extra curvature in her bottle not represented in the given graph. For 2a and 2b, her graphs are potentially missing curvature in her graph to represent the inward curvature underneath the cap of the bottle. We return to 2b, as does Kala, in her interview.

In analyzing Kala's submitted work (Figure 3) and video from Part 2, we note three aspects. First, in Part 2: Problem 1a, Kala highlights a single upward curvature in her graph with a corresponding pink cross section of her bottle in which the height has *both* an increasing and decreasing rate of height with respect to volume. In describing the pink section in her graph, she stated it is "dip[ping] down" and "the volume is decreasing, while the height is still increasing." When relating that to the pink section in her bottle, she stated "because the volume is decreasing, so the bottle gets skinnier at that part". Similar work is in Part 2: Problem 2 (e.g., green section). Thus, we infer Kala associated a thinner portion of a bottle with a "dip" in her graph.

Second, Kala uses unit squares to represent equal unit cubes of volume. For example, in Part 2: Problem 1c, in which the volume of the bottle increases by two units for each additional inch in height, Kala created a table of volume values and then drew a corresponding unit cube sketch using her table values. She described her construction of this "cube diagram" as follows: "So we start with two, we do for our one inch of height, we have two blocks of volume. And then two inches, we have a total of six." The two and six correspond to the volume values in her table for a height of one and two inches, respectively. Similar work is in Part 2: Problem 2.

Third, after plotting points and creating sketches using unit squares using tables she constructed, she inconsistently connects the points and creates final bottles. For instance, Part 2: Problem 1b does not have a smooth curve in her graph, whereas Part 2: 1c does. Similarly, Part 2: Problem 1b does not have a smooth curve in her final bottle, whereas Part 2: 1c does. Her interview provides more details on how she chose to connect points on her graph.

Kala's Interview on the Bottle Problem

Figure 4 is Kala's final work on the Bottle Problem (left) and her additional work done throughout the interview on the Bottle Problem (right). Note that the bottle is the same one from

Part 1: Problem 2b (Figure 1) and that the resulting graphs are very similar. However, new for the interview was her use of a "cube diagram", a table of values, and plotted points (methods she used during Part II (Figure 3)). In this section, we focus explicitly on Kala's CR given (i) we have insights into her methods for constructing graphs from the Bottle Problem assignment and (ii) the interview enabled more in-depth questioning given its semi-structured nature.

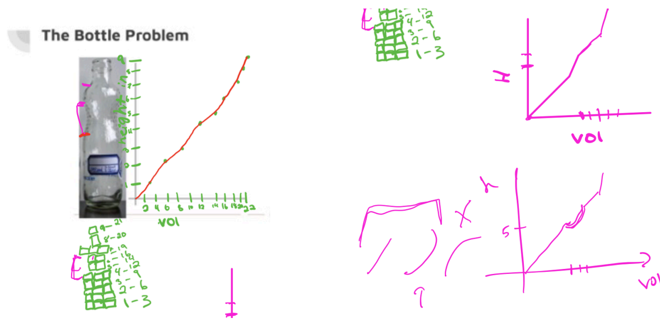


Figure 4: Kala's final work on the Bottle Problem during her cognitive interview.

First, we note Kala's quantification of volume and height using a diagram of the cross section of the bottle. Her use of her "cube" method to quantify volume enables her to draw a diagram in which she can measure volume via counting the cubes. She also defines "layers" of cubes, each layer corresponding to one inch of height. In the green "cube diagram" in Figure 3, Kala uses this diagram to measure the volume in the bottle for each additional inch of height (e.g., "1-3" indicates at 1 inch of height, the volume is 3 unit cubes, and then 2-6, 3-9, 4-12, 5-14, and so on).

Second, we note that Kala attempts to describe how her two quantities, height and volume, are changing together in her graph. For instance, she describes the initial straight segment emanating from the origin (in all her graphs, but her final graph, specifically) the following way: "what this is kind of showing that it's increasing at the same time, as the height is increasing, the volume is increasing at the same time". This corresponds to the second level of CR.

Third, we note that Kala recognizes that the "dips" in the cross section of the bottle impact how much volume can fit within a single layer. For instance, when describing what happens after the straight section of the bottle, she states "the height of the water is gonna increase, and the volume-there's less room for water." She uses that observation to justify why her *graph* should not continue to be straight at these "dips". She states, "the height is staying the same but the volume is gonna decrease, because there's less room so we kind of dip-kind of like dip a little bit and then come back up." Here, although she states the *volume* is going to decrease, she is referencing her layers in her bottle, noting that the layers are going from three unit cubes down to two unit cubes. She also clarifies later that it was the one inch increment that "is staying the same" for the height and that "the height is still increasing as we go up, since we're moving up the bottle since we are filling it up." Thus, though not explicitly using *amounts of change* language, we see Kala attending to and comparing changes in height and changes in volume.

Lastly, we note the perturbation Kala experienced when trying justify straight segments versus curved segments. To discuss this perturbation, we focus on when Kala is referencing the bottle from four to six inches, which corresponded to the pink region labeled on the given cross section of the bottle (i.e., the "dip"). In her cube diagram, this region corresponded to two layers of two unit cubes each. In her graph, this corresponded to the curve immediately after the straight portion of her graph (see bolded region and the three curves she considered in Figure 4 (right)).

When considering using straight lines in her graph, her reasoning is as follows:

Kala: So we're at five inches, which is where that fifth layer shrinks to two, and then I have fourteen cubic whatever is of volume, and then the volume, the height is still increasing, so we jump to six, but it's only fourteen units of volume.

From this quote, we see that Kala identified that there is less volume gained per inch of height in the layers (that each have two cubes) than the first four layers (that each have three cubes). She concluded that the line segment corresponding to that "dip" in the bottle has a steeper slope than the line segment corresponding to the first section. This use of "increasing rate" implying "steeper" straight line segment is what Stevens (in press) described.

When considering curved lines, she hesitated to associate it only with the curve of the bottle, referencing her bottle from Assignment 3: Part 1 2a (Figure 2), where the curvy bottle still resulted in a graph with a straight line. She stated she felt like the bottle was "trying to trick me and make me want it [the graph] to be curvy" because "the bottle is curvy". In the end, she chose her final graph to have a curved segment in that region vs. a straight segment, though admitting, "So I honestly don't know which is the move-but let's go curvy".

In summary, Kala wanted to justify a "curvy" graph, but given her reasoning with the bottle's cross section, she could not. In the discussion, we discuss why Kala's reasoning with the diagram was both quantitative and limited, and its implications for students' *amounts of change* CR.

Discussion and Implications

Across Kala's work, we noted ways of reasoning about *directional change* but a struggle to reason about rates of change relative to her peers. From her PCA results and assignment, there was not much evidence about how she determined rates of change, only that she seemed to connect the idea to "dips" in the bottle in the Bottle Problem. However, in her interview, where she used her quantitatively appropriate "cube diagram" method, we find Kala's *amounts of change* reasoning had one major limitation: she chose height intervals that spanned across two different rates of change (i.e., decreasing rate then increasing rate). This decision resulted in *equal* changes in volume for a successive change in height in her cube diagram given the symmetry of the dip in the bottle. Because this constant change matched the constant changes in volume she identified for successive changes in height for the straight segment of the bottle, and because that constant change in volume is how she justified a straight line segment in her graph, she concluded that the "dip" should do the same (but steeper to accommodate less volume added). However, this conclusion perturbed Kala, who still drew a curve on her graph.

To justify appropriate curvature, Kala would have needed to identify smaller amounts of change in height, successive changes in height that started at the beginning of the inward curvature of the bottle and symmetric with the outer curve of the bottle (e.g., Figure 5). Understanding the necessity of this careful construction of *amounts of change* is crucial for making appropriate quantitative conclusions about the CR of the quantities. From Kala's work, we see the importance of supporting this intellectual need for the careful construction of *amounts of change*. This need goes beyond a need to compare amounts of change in one quantity for equal successive changes in another quantity, because as Kala illustrated, she had that goal.

Figure 5: Equal changes in height resulting in (left) equal vs (right) decreasing/increasing changes in volume.

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