Guided Reinvention of the Definitions of Reducibles and Irreducibles

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Informed by Realistic Mathematics Education, we designed a hypothetical learning trajectory on graduate students' guided reinvention of reducible and irreducible elements in unique factorization domains. We created experientially real tasks for use in a teaching experiment, in which students used algebra tiles as an emergent model of factoring integers and quadratics in $\mathbb{Z}[x]$. *In students' mathematical activity, this became a model for abstracting the shared structure of (ir)reducible elements in* $\mathbb{Z}$ *and* $\mathbb{Z}[x]$, *which students used as they formally defined (ir)reducibles.*

Keywords: Realistic mathematics education, abstract algebra, hypothetical learning trajectory

Abstract algebra courses that serve in-service and prospective teachers (IPSTs) should guide IPSTs to reason about the advanced content in a way that helps them reshape their understanding of secondary mathematics (Wasserman, 2018). Abstract algebra can be beneficial for helping teachers develop unified understandings of algebraic concepts that span K-16 curricula, such as the factorization of integers and polynomials (e.g., Murray et al., 2017; Novotná & Hoch, 2008; Usiskin, 1974). The Common Core State Standards of Mathematics (National Governors Association & Council of Chief State School Officers, 2010) suggested that teachers should teach students to "understand that polynomials form a system analogous to the integers" (p. 64). An abstract algebra course can support IPSTs in recognizing properties shared by the sets of integers ($\mathbb{Z}$) and polynomials with integer coefficients ($\mathbb{Z}[x]$). $\mathbb{Z}$ and $\mathbb{Z}[x]$ are *unique factorization domains* (UFD), defined as an integral domain R in which every nonzero element $r \in R$ that is not a unit has the following two properties: (i) r can be written as a finite product of irreducibles p_i of R (not necessarily distinct), $r = p_1 p_2 \dots p_n$. (ii) The decomposition in (i) is unique up to associates: namely, if $r = q_1 q_2 \dots q_m$ is another factorization of r into irreducibles, then $m = n$, and there exists some renumbering of factors so that p_i is associate to q_i for $i = 1, \dots, n$ (Dummit & Foote, 2004). Reasoning about UFDs could help IPSTs better understand the properties related to factorization of elements in $\mathbb{Z}$ and $\mathbb{Z}[x]$, e.g., the fundamental theorem of arithmetic and the fundamental theorem of algebra. Before one can understand this structure of UFDs, they need to understand the structure of irreducible and reducible elements. An *irreducible* is a nonzero, non-unit element r of an integral domain R such that if $r = ab$ with $a, b \in R$, then a or b must be a unit in R. A *reducible* is a nonzero, non-unit element r of an integral domain R such that we may write $r = bc$ for $b, c \in R$ and neither b nor c is a unit.

There has not yet been research done on how IPSTs can define the shared structure of factorability of the elements in $\mathbb{Z}$ and $\mathbb{Z}[x]$. We address this by investigating how graduate student IPSTs reinvent the definitions of irreducibles and reducibles. We used the instructional design theory of Realistic Mathematics Education (RME; Freudenthal, 1991; Gravemeijer, 1999; Gravemeijer & Doorman, 1999) to design a hypothetical learning trajectory (Simon, 1995) of how IPSTs can reinvent the definitions of reducibles and irreducibles. This is part of a larger project in which we designed a local instructional theory (Gravemeijer, 2004) guiding IPSTs to reinvent UFDs in connection to their teaching. We address the following research questions: *Which aspects of the task sequence support IPSTs' reinvention of (ir)reducibles? Which ways of reasoning are productive to leverage to guide IPSTs toward reinventing (ir)reducibles?*

Literature Review

Reasoning about the properties of abstract algebraic structures can help IPSTs develop more robust understandings of the secondary algebra content they teach. Researchers have shown how reasoning about the properties of algebraic structures can help students or IPSTs develop unified understandings (Lee & Heid, 2018; Zandieh et al., 2017) of secondary mathematics content. Having a *unified understanding* of a concept involves recognizing the shared overarching structure of the concept among different instantiations of that concept in different contexts and being able to reason about that concept generally without needing to reference a specific context. Cook et al. (2023) identified students' unified ways of reasoning about the structure of inverses shared by different instantiations of that inverse concept. Serbin (2023) showed how reasoning about the group axioms supported IPSTs in recognizing the shared overarching identity structure shared by additive, multiplicative, and compositional identities. Several other researchers have explored students' unified understandings of algebraic concepts, including homomorphisms (Melhuish et al., 2020), linear transformations (Bagley et al., 2015; Zandieh et al., 2017), and binary operations (Wasserman, 2023). These studies illustrate how students and IPSTs reason about the shared structure of various algebraic concepts in different contexts.

We aimed to guide IPSTs to recognize the structure shared by certain types of integers and polynomials, those that are composite or factorable (reducibles) and those that are prime or not factorable (irreducibles). There has been minimal research done on students' understanding of reducible and irreducible elements. Two studies (Lee, 2018; Lee & Heid, 2018) explored the coherence of students' understandings of the factorization of integers and polynomials as instantiations of the decomposition of elements in integral domains. Lee (2018) investigated how students developed a unified understanding of factoring across elementary, secondary, and tertiary contexts. The students who perceived factoring whole numbers and factoring polynomials as disparate procedures could later recognize the shared structure of factorization of elements in integral domains. Juxtaposing integers and polynomials allowed the students to reflect on parallel structures within the sets, such as primes and irreducibles, composites and reducibles, and prime factorization and irreducible factorization. The students developed unified understandings of the shared structure of integers and polynomials as integral domains in which not every element had a multiplicative inverse. Lee and Heid's (2018) study informed our study design that guides IPSTs to identify the shared structure of primes and irreducible polynomials, as well as composites and reducible polynomials, to reinvent irreducibles and reducibles.

Theoretical Background

Design Research (Gravemeijer et al., 2003) involves a cycle of designing sequences of mathematical tasks to be used during instruction, implementing the task sequences, analyzing students' learning, and revising the task sequences based on the student learning outcomes. In designing the task sequence, design researchers develop a Hypothetical Learning Trajectory (HLT), which consists of the learning goal, the learning activities, and the hypothesized thinking and learning that students might engage in as they perform the activities (Simon, 1995). Through the process of designing, implementing, and revising HLTs, researchers can discover students' ways of thinking that anticipate formal mathematics and identify ways to leverage those ways of thinking to guide students' development of formal concepts. Our design of an HLT is guided by the instructional design theory of RME (Gravemeijer, 1999), which conceptualizes mathematics as a human activity. The HLT is designed based on the three RME heuristics: 1) experientially real tasks, 2) emergent model, and 3) guided reinvention. Experientially real tasks allow the students to connect with their informal knowledge, developing an intuitive understanding that will help them create an emergent model for use in that given context, defined as a *model-of* their situated activity. The emergent model shifts from being *model-of* a student's activity to a *model-for* formalizing mathematics as students mathematize their mathematical activity (Zandieh & Rasmussen, 2010). *Horizontal mathematization* involves "describing a context problem in mathematical terms to be able to solve it with mathematical means" (Gravemeijer & Doorman, 1999, p. 117), and *vertical mathematization* involves "mathematizing one's own mathematical activity" (p. 117). Design researchers create experientially real tasks that require horizontal and vertical mathematizing by which students, with guidance, can reinvent mathematical concepts.

Mathematizing occurs as students engage in mathematical activity, such as defining and conjecturing (Rasmussen et al., 2005). Defining is a process of formulating, negotiating, and revising a definition (Zandieh & Rasmussen, 2010). Freudenthal (1973) described two types of defining: descriptive and constructive. In *descriptive defining,* students build a definition by suggesting ideas about an object's properties, and the instructor helps organize and clarify such ideas. *Constructive defining* involves students modeling a new object by using prior examples or ideas. Rasmussen et al. (2005) deemed descriptive defining as horizontal mathematizing, in which the student organizes their ideas into a structured definition, and constructive defining as vertical mathematizing, where students abstract and generalize previously organized activities to create a new concept. Descriptive defining (horizontal mathematizing) grounds the context for students to progressively advance their activity to constructive defining (vertical mathematizing).

Methods

We designed an RME-based HLT with the goal of guiding students to reinvent the definitions of reducibles and irreducibles. We created "experientially real" tasks related to a high school/college algebra context that used algebra tile manipulatives. These tasks were designed to help the IPSTs develop an intuitive understanding of the mathematical concepts and engage in mathematical practices of defining and conjecturing, which would support them in the process of formalizing them. We made conjectures of students' reasoning for each of the designed tasks in our task sequence. We tested this HLT by administering tasks to IPSTs in a teaching experiment (Steffe & Thompson, 2000) conducted in a Hispanic-serving institution in the Southern US. Six mathematics graduate student IPSTs participated in this study. They all took Abstract Algebra courses at the undergraduate or graduate levels but have not learned UFDs in their coursework. They were grouped into three pairs (Group A with Josie and Raul, Group B with Javier and Roberto, and Group C with Kim and Taylor). Groups A and B participated in person, and Group C participated on Zoom. In-person groups were given physical algebra tiles, and the virtual group was provided with an online whiteboard and virtual algebra tile manipulatives.

The research team consisted of a teacher-researcher (TR), secondary instructor, and research assistant. The TR's role was to guide the session, provide students with prompts to work on, and ask follow-up questions to better understand and clarify the students' thinking. The research assistant operated the video camera and took field notes. The secondary instructor observed, took field notes, and asked questions to students when needed. We collected and transcribed the video recordings of fifteen 1.5-hour sessions of the teaching experiment (five sessions per group). The groups' written and virtual whiteboard work was collected. We used inductive coding (Miles et al., 2013) to analyze participants' reasoning, mathematizing, and the development of emergent models as they used algebra tiles in the task sequence. We focused our analysis on students' ways of reasoning that were leveraged by the TR to guide them to reinvent (ir)reducibles. In what follows, we describe the design of each task set in the HLT, how they were informed by RME, and the high-leverage student reasoning (italicized) evident in students' responses.

Results

Task Set 1: Contrasting Composite and Prime Integers

The experientially real tasks in Task Set 1 (see Figure 1a) were designed to help students develop an intuitive understanding of the factorability of integers as they use the emergent model of algebra tiles. The students created a *model of* factoring irreducibles and reducibles as they manipulated algebra tile representations of integers or quadratics in $\mathbb{Z}[x]$. Tasks guided students to see some integers, like 12 represented by 12 one-unit-squared tiles joined together, can be arranged into rectangular arrays where its factors are the side lengths of the rectangle. The first way of reasoning evoked was that *the factors of an integer correspond to the possible side lengths of the rectangular array that represents the integer*. This is evident in group C:

> *Kim*: My thought process is just like, how are all the different ways that…you can factor 12, like with different, with smaller decompositions of numbers.
>
> *TR*: … how does the factor translate to this rectangle concept here?
>
> *Taylor*: It's length times width, right? That's what I would think of it. You know, 3 times 4 and 4 times 3, and 1 times 12, and 12 times 1, and 6 times 2, and 2 times 6.

When Kim and Taylor worked on problem 1, they recognized that 12 could be represented in different rectangular arrays that each had side lengths of the different pairs of factors of 12.

This led to their second high-leverage idea was that *the rectangle that was 12 units* $\times$ 1 *unit was essentially the same as the rectangle with dimensions of 1 unit* $\times$ 12 *units due to the commutativity of integer multiplication*. This is evident in this transcript from group A:

> *Josie*: Do these count as two separate ones?
>
> *TR:* That's a great question. So you're saying, 1 by 12, is that the same as 12 by 1?
>
> *Josie*: …Well, I figured…multiplication is commutative, so they're exactly the same.

Josie and Raul thought that two rectangles with dimensions from the same factor pair were essentially the same, so they did not count as two different rectangles. This idea was used in the next task, 2a (see Figure 1), which was designed to help students recognize that prime numbers of square tiles can have only one arrangement that forms a rectangular array. Josie explained:

> Numbers that can only be written in one way would be the prime numbers. They don't have too many factors. They only have that one and itself. And if we're only counting like the- the 1 by 5 as the same as 5 by 1- that's really the only way you can do it.

Josie used the aforementioned way of reasoning that rectangular arrays with the same dimensions counted as the same representation to justify her claim that *numbers that could only be represented in one rectangular array must be prime*. This claim is the third way of reasoning that could be leveraged toward guiding the students to reinvent reducibles and irreducibles. Once the students recognized that those types of integers were prime, they used that way of reasoning in task 2c to identify the difference between the integers that could only be represented in one rectangular array and the integers that could be represented in many rectangular arrays. Josie responded, "The amount of factors that they have. Primes only have 1 and the number itself, and composites have at least 2." This elicited the way of reasoning that *the difference between the integers that could only be represented in one rectangular array and the integers that could be represented in many rectangular arrays was whether they were prime or composite*. This is the first instance where they use the emergent model (the algebra tiles) as a *model for* identifying the differences between irreducible and reducible elements in $\mathbb{Z}$. Overall, these tasks were designed to prompt the students to contrast the primes and composites using the emergent model. We posit that making this contrast is an essential waypoint in guiding students to reinvent (ir)reducibles.

a) Task Set 1	1. Find all the possible ways that the number 12 can be represented using rectangular arrays of squares. 2. Like the number 12, some numbers can be written in many different ways using rectangular arrays of squares. But some numbers can only be written in one way. a) Find several numbers that can only be written using one rectangular array of squares. b) Find several numbers that can written using different rectangular arrays of squares. c) What is the difference between these two kinds of numbers?
b) Task Set 2	1. Try to arrange these quadratics in a rectangular array using algebra tiles. Each side length of a rectangular array cannot be a constant number and must include the variable x. a) $3x^2 + 9x$ b) $x^2 + 3x + 3$ c) $2x^2 + 4x$ d) $x^2 + 4x + 3$ e) $x^2 + 5x + 7$ 2. What is the difference between the quadratics you **can** arrange into a rectangular array and the quadratics you **cannot** arrange into a rectangular array? Make a conjecture. 3. Give 3 examples of quadratics whose algebra tile representations could be arranged into rectangular arrays. 4. Give 3 examples of quadratics whose algebra tile representations could NOT be arranged into rectangular arrays. 5. Does your conjecture from Task 2 about the arrangement of the algebra tiles hold true for these examples? 6. You conjectured that: *A quadratic can be arranged into a rectangular array with algebra tiles if and only if the quadratic is factorable*. Let's test this conjecture with the following quadratics. a) $x^2 - 4$ b) $x^2 - 2$ c) $x^2 + 4$ 7. Refine your conjecture: *Quadratics can be arranged into a rectangular array with algebra tiles iff:*
c) Task Set 3	1. What similarities can you see between the integers that can be represented in different rectangular arrays and the quadratics that can be represented as a rectangular array with algebra tiles? 2. What similarities can you see between the integers that can only be represented in one rectangular array and the quadratics that cannot be represented as a rectangular array with algebra tiles? 3. We can name the first type of integers and quadratics that can be represented as a rectangular array as "**reducibles**." We can name the second type of integers that can only be represented in one rectangular array and quadratics that cannot be represented in any rectangular array as "**irreducibles**." We will formally define these terms soon. When irreducible integers can be arranged into a rectangle, one of its side lengths must be 1 or –1. These numbers are called "**units**." Definition: An element of a ring is called a unit if it has a multiplicative inverse. In a ring, an element is either 0, a unit, an irreducible, or a reducible. What are the units of $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients? 4. We can name the first type of **integers** and **quadratics** that can be represented as a rectangular array as "reducibles." Create a definition for a reducible. 5. We can name the second type of integers that can only be represented in one rectangular array and quadratics that cannot be represented in any rectangular array as "irreducibles." Define irreducible.

Figure 1. The three task sets in the HLT's task sequence.

Task Set 2: Contrasting Reducible and Irreducible Quadratics in $\mathbb{Z}[x]$

The tasks in Set 2 guided students to use the intended emergent model of algebra tiles as a *model of* factoring quadratics in $\mathbb{Z}[x]$ by attempting to arrange quadratics into rectangular arrays. The students recognized $3x^2 + 9x$, $2x^2 + 4x$, and $x^2 + 4x + 3$ can each be arranged into a rectangular array, where each side length of a given array is a factor of the quadratic, whereas $x^2 + 3x + 3$ and $x^2 + 5x + 7$ could not be arranged into a rectangular array (see Figure 2). The tasks had them find the difference between these quadratics and guided students to conjecture *(a) quadratics that can be factored can be arranged into a rectangular array where the factors of the quadratic are the side lengths of the rectangle and (b) the quadratics that cannot be factored cannot be arranged into a rectangular array.* The conjectures are high-leverage ways of reasoning that can be used to support students in reinventing (ir)reducibles. Kim used these:

> For the quadratics that you can arrange in a rectangular array, the quadratic is factorable, … if you have a quadratic you cannot arrange into a rectangular array, then…it is not factorable… if the quadratic is factorable, then you can arrange it in a rectangular array.

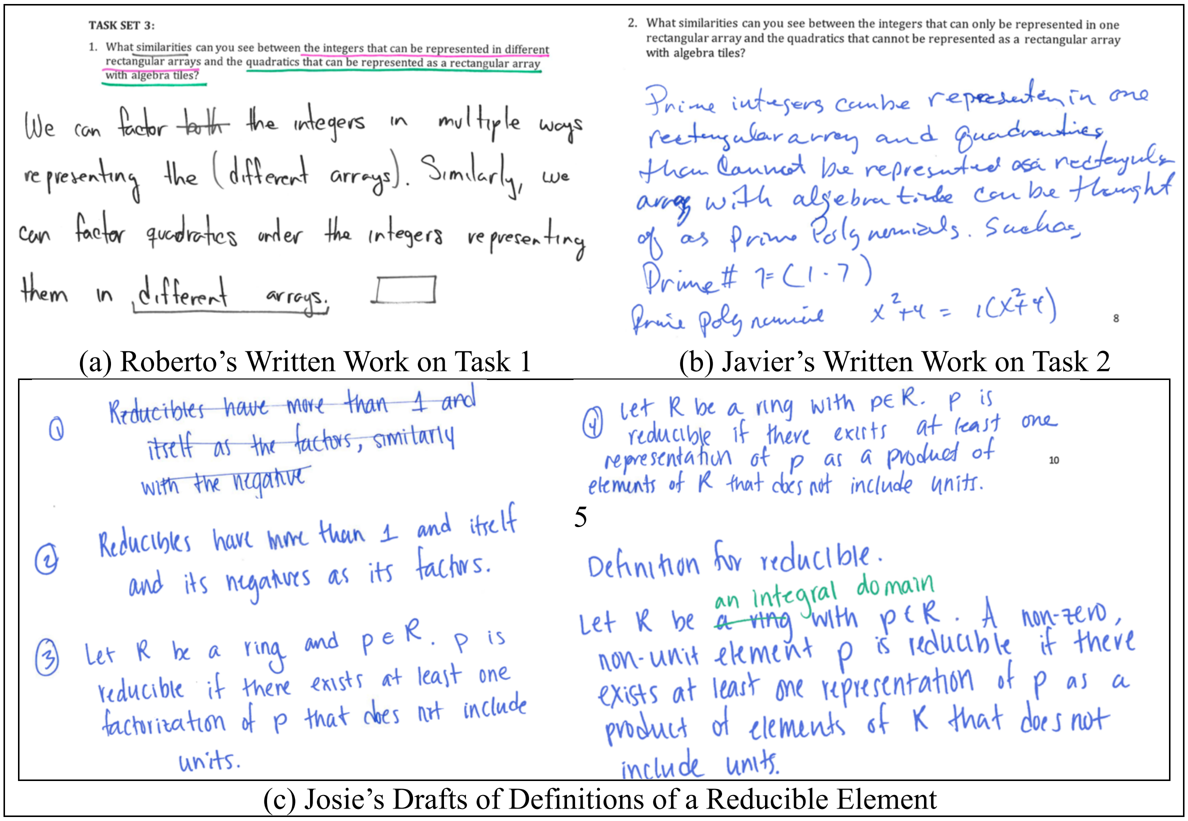
This discussion led group C to refine their conjecture to be "A quadratic can be arranged into a rectangular array with algebra tiles if and only if it is factorable." The tasks guided students to consider quadratics that are factorable over $\mathbb{R}$ but not over $\mathbb{Z}$, which cannot be arranged into a rectangle, which led them to revise their conjecture to specify the set over which the quadratic was factorable: *A quadratic can be arranged into a rectangular array with algebra tiles if and only if the quadratic is factorable over the integers*. These tasks guided the students to contrast irreducible and reducible quadratics in $\mathbb{Z}[x]$ to help them develop intuitive understandings of the difference between them in this set. Making this contrast is a high-leverage way of reasoning that is an essential waypoint in guiding students to reinvent the definitions of (ir)reducibles.

Task Set 3: Comparing and Defining Reducibles and Irreducibles in $\mathbb{Z}$ and $\mathbb{Z}[x]$

The tasks in Set 3 (see Figure 3) were designed to guide students to abstract and generalize the structure shared by reducible elements in $\mathbb{Z}$ and $\mathbb{Z}[x]$, as well as the structure shared by irreducible elements in $\mathbb{Z}$ and $\mathbb{Z}[x]$. The tasks prompted students to identify similarities among the modeled representations of the integers and quadratics in $\mathbb{Z}[x]$ that could be arranged into rectangular arrays with non-unit side lengths. Roberto responded, "They're factorable in the integers… one of them is composite, so it can be written in different ways, and the quadratics can be represented as a rectangular array." Prompting students to identify the similar structure of the modeled composite integers and quadratics allowed students to recognize that *integers and polynomials of this type can be factored or reduced into a product of integers or polynomials of lesser degree*. This way of reasoning about the reducibility property of these types of integers and quadratics can be leveraged to guide the students to reinvent the definition of reducibles.

In task 2, the students were asked to identify similarities among the modeled representations of the integers and quadratics in $\mathbb{Z}[x]$ that could not be arranged into rectangular arrays with non-unit side lengths. Javier responded, "Prime integers can be represented in [one] rectangular array. Like when 7 can be 1 times 7… quadratics that cannot be represented as a rectangular array with algebra tiles can be thought of as prime polynomials, such as $x^2 + 4$" (see written work in Figure 3). Prompting students to identify the similar structure of the modeled prime integers and quadratics led them to recognize that *prime integers and quadratics of this type have the same characteristic of not being factorable into a product of integers or polynomials of lesser degree*. This way of reasoning about the irreducibility of these types of integers and quadratics can be leveraged to guide students to reinvent the definition of irreducibles. These tasks intended to evoke a transition in the students' emergent model (i.e., the algebra tiles), provoking the students to use their *model of* factoring integers and quadratics in $\mathbb{Z}[x]$ as a *model for* abstracting the shared structure of reducibles and irreducibles, which could be used to define those concepts.

Before the tasks could guide students to formally define (ir)reducibles, the students needed to understand the concept of a unit in a ring because the definitions refer to a factor being either a unit or not. Task 3 gave the students the definition of a unit and prompted them to identify the units of $\mathbb{Z}$ and $\mathbb{Z}[x]$, which are 1 and -1 in both sets. Once the students had this terminology of "unit" they could use in their definition drafts, tasks 4 and 5 guided students to leverage their prior conjectures to give rough draft definitions of reducibles and irreducibles. The students iteratively refined their definition drafts with support from the TR's pedagogical moves of scaffolding, poking holes in arguments, posing counterexamples, pressing for precision, encouraging the definitions to be nonredundant and generalizable, and making sure any example of the concept satisfied the definition. Josie's five drafts of definitions of reducibles, which progress from using informal to more formal mathematical terminology, are shown in Figure 4. Overall, task set 3 were designed to engage students in constructive defining and vertical mathematizing, i.e., further mathematizing their previous conjectures about properties of reducibles and irreducibles in pursuit of reinventing and defining reducibles and irreducibles.



(a) Roberto's Written Work on Task 1
(b) Javier's Written Work on Task 2
(c) Josie's Drafts of Definitions of a Reducible Element

Figure 3. Students' written work in Task Set 3.

Discussion and Conclusion

This study shows how the task sequence and TR's pedagogical moves guided the students' reinvention of the definition of reducibles and irreducibles by facilitating their high-leverage reasoning about reducible and irreducible properties of integers and polynomials. The sequence of three task sets prompted students' reasoning in identifying and abstracting the reducibility/irreducibility of integers (Task Set 1) and quadratics in $\mathbb{Z}[x]$ (Task Set 2). In particular, the experientially real tasks using algebra tiles evoked students' development of emergent models *of* factorizing integers and quadratics, which transitioned to become models *for* identifying and refining their conjectures about the shared structure of integers and quadratics (Task Set 3). In this process, TR's pedagogical moves played an essential role in developing the emergent models and facilitating students' mathematizations (Gravemeijer & Doorman, 1999) by helping them to make sense of their mathematical activities of creating and interpreting algebra tile models (horizontal mathematization) and by guiding them to formalize their conjectures of the shared structure of integer and quadratics in their definitions of reducibles and irreducibles in an integral domain (vertical mathematization). This finding suggests a hypothetical learning trajectory of students' reinvention of definitions in advanced mathematics by eliciting and leveraging students' reasoning in experientially real tasks. Future studies can extend the findings of this study by refining the task design and implementation and by examining different types of high-leverage reasoning that can also guide students in learning trajectories.

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References

Bagley, S., Rasmussen, C., & Zandieh, M. (2015). Inverse, composition, and identity: The case of function and linear transformation. *The Journal of Mathematical Behavior*, *37*, 36–47.

Cook, J. P., Richardson, A., Reed, Z., & Lockwood, E. (2023). Using conceptual analyses to resolve the tension between advanced and secondary mathematics: the cases of equivalence and inverse. *ZDM–Mathematics Education*, 1–14.

Dummitt, D.S. and Foote, R.M. (2004). *Abstract algebra* (3rd ed.). John Wiley & Sons, Inc.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Netherlands.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, 6(2), 105–128.

Gravemeijer, K., Bowers, J., & Stephan, M. (2003). Chapter 4: A hypothetical learning trajectory on measurement and flexible arithmetic. *Journal for Research in Mathematics Education. Monograph*, *12*, 51–66.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, *39*(1-3), 111–129.

Lee, Y., & Heid, M. K. (2018). Developing a structural perspective and its role in connecting school algebra and abstract algebra: A factorization example. In *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers* (pp. 291–318). Cham: Springer.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *The Journal of Mathematical Behavior, 60*, 100806.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2013). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Thousand Oaks, CA: Sage Publications.

Murray, E., Baldinger, E., Wasserman, N., Broderick, S., & White, D. (2017). Connecting advanced and secondary mathematics.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Washington, DC: Authors. http://www.corestandards.org/Math/.

Novotná, J., & Hoch, M. (2008). How structure sense for algebraic expressions or equations is related to structure sense for abstract algebra. *Mathematics Education Research Journal*, 20(2), 93–104.

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 51–73.

Serbin, K. S. (2023). Prospective teachers' unified understandings of the structure of identities. *The Journal of Mathematical Behavior*, *70*, 101066.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114–145.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education* (pp. 267-306).

Usiskin, Z. (1974). Some corresponding properties of real numbers and implications for teaching. *Educational Studies in Mathematics*, 279–290.

Wasserman, N. H. (2018). Knowledge of nonlocal mathematics for teaching. *The Journal of Mathematical Behavior*, *49*, 116–128.
Wasserman, N. H. (2023). Mathematical Challenge in Connecting Advanced and Secondary Mathematics: Recognizing Binary Operations as Functions. In *Mathematical challenges for all* (pp. 241-260). Cham: Springer International Publishing.
Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior*, *29*(2), 57–75.
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics*, *95*(1), 21–38.