The Problems with "Any" and "All": How Language Impacts Student Meanings for Quantified Variables

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This study investigates undergraduate students' meanings for quantified variables, and how the language of a mathematical statement impacts their meanings. I conducted 3-day exploratory teaching interviews with eight students from transition-to-proof and advanced calculus courses (four from each course). Over the course of the three days, I presented students with a variety of statements from calculus in order to test how both and language and mathematical content might impact students' meanings for quantified variables. I found that students' meanings did often change across mathematical contexts that contained different language. In particular, the words "any" and "all" were found to lead some students to use different meanings for quantified variables in calculus statements.

Keywords: quantification, quantifiers, meaning, calculus, mathematical language

Recently, many have called others to focus on students' logic in elementary calculus courses (Case, 2015; Dawkins & Cook, 2017; Sellers et al., 2021). This is because many calculus statements are *complex mathematical statements* (Sellers, 2018), i.e., involving quantified variables and logical connectives. Yet, when investigating student quantifications of individual quantified variables, some students, in particular calculus students, use different quantifications for individual quantified variables, if they quantify at all (Sellers et al., 2021).

Furthermore, other researchers have called for more investigation into students' meanings for quantified variables as related to the specific language and grammar given in statements across different mathematical domains (Shipman, 2013; Vroom, 2022). As such, students' quantifications of the variables in these statements as related to the specific language given in the statements is important to investigate if we want to help them understand calculus ideas.

Others have noted that students' logic might vary from one mathematical context to the next (Dawkins & Roh, 2020; Dubinksy & Yiparaki, 2000; Epp, 1999; Shipman, 2013). Thus, there is a need to compare students' meanings for quantified variables across different mathematical contexts which involve different language and grammar.

This study is part of a larger study investigating both students' interpretations for quantified statements from calculus and their negations of these statements (Sellers, 2020). In this paper, I focus on additional information gleaned regarding students' meanings for quantified variables as related to the language given across different statements. Specifically, I focus on findings that extend the work of Sellers et al.'s (2021) framework for students' meanings for quantified variables and continue to address the following research question: *How does language impact student quantifications across different mathematical contexts in calculus?*

Literature Review

Several researchers have noted the complexities of interpreting quantifiers specifically in complex mathematical statements (David et al., 2020; Dubinsky et al., 1988; Selden & Selden, 1995; Sellers et al., 2021). Furthermore, some have posited that interpreting quantified statements may be inherently difficult for students due to the different purposes for quantifier words and logical connectives in mathematics and colloquial English (Epp, 1999; Dawkins &

Cook, 2017; Grice, 1975; Roh, Lee, & Tanner, 2016). For example, if I state the following universally quantified statement, “Every book on the shelf is French,” I colloquially infer there is at least one book on the shelf, whereas the statement could be vacuously true in propositional logic (Epp, 1999; Johnson-Laird, et al., 1989). Even more striking, there may be cases where the meaning of quantifier may be completely different in everyday language than in mathematics. Shipman (2013) concluded that students thought “unique” meant “unequaled” instead of “sole,” and conjectured that her students’ use of the word may have been attributed to the colloquial use of the word. Roh, Lee, and Tanner (2016) found that students assumed the article “a,” attached to the existential quantifier “there is,” referred to *one* single quantity.

While these previous works do suggest that colloquial language impacts students’ reasoning in the mathematics classroom, I do not mean to suggest that instructors should utilize formal semantics or symbolism in the presentation of logical rules exclusively. Stylianides, et al. (2004) found that elementary majors’ ability to reason about the truth of statement was higher in verbal cases than symbolic ones. Thus, verbal representations of mathematical ideas are not necessarily to be avoided; rather, an awareness of these colloquialisms may help teachers address these inconsistencies in natural language and mathematics as the need arises in their classrooms.

Theoretical Perspective

The findings from this study use and extend Sellers et al.’s (2021) framework for students’ meanings for quantified variables. Similar to Sellers et al. (ibid), I approach this study from a constructivist (Glasersfeld, 1995) perspective. Thus, my focus is on *individual* students’ meanings for quantified variables in a variety of calculus statements. A meaning is a scheme, and resides in a student’s mind (Piaget, 1977; Thompson, et al., 2014).

Sellers et al.’s framework for student quantifications. In Sellers et al.’s (2021) framework for students’ meanings for quantified variables (i.e., quantifications), the authors previously identified five different quantifications. Each of meanings for quantified variables (i.e., quantifications), abbreviated MQ1-MQ4 and NQ, are detailed below. MQ1-MQ4 are *not* intended to be developmental; they are only numbered for convenience of discussion.

MQ1 and MQ2 are indicative of meanings similar to conventional meanings for “there exists and x” and “there exists a unique x,” respectively. Students using MQ3 in a specified moment chose a value for x from their chosen domain of discourse and determined whether or not their chosen value for x satisfies the predicate. They then repeated (or imagined repeating) this mental action by checking the predicate for *multiple* values of x until they exhausted *all* the elements in the domain of discourse.

While MQ1-MQ3 are akin to mathematical meanings for the existential, existential unique, and universal quantifiers, MQ4 does not align with mathematical convention. Students using MQ4 chose or looked at a value of x and then checked the predicate only for this selected value. These students sometimes chose another value of x and checked the predicate for this other value, but these values were not strategically selected, but *spontaneously* selected.

Finally, there were other moments where students did not quantify at all, and thus were described as using no quantification (NQ). These students appeared to search for the domain of discourse for a variable, but they did not appear to check the predicate for any values of x. Students that used NQ did not state that they are searching for a specific number of elements of x to satisfy the predicate, but instead interpreted the statement as if it were true. Furthermore, these students simply confirmed what they think the predicate means to them, rather than checking whether or not the values satisfy the predicate.

Methods

In order to further investigate students' meanings for quantified variables in calculus statements, I interviewed four students currently in transition-to-proof courses and four students from advanced calculus or above as part of a larger study.[1] I chose these students because they had *already completed* an entire calculus sequence, and had thus, been exposed to all of the necessary concepts in the statements. These students were selected from a pool of 24 students based on a survey used for theoretical sampling (Patton, 2001). Selected students participated in three exploratory teaching interviews (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020; Thompson et al., 2014).

Each exploratory teaching interview (ETI) lasted approximately 1-3 hours each, for a total of 5-8 hours, depending upon student responses. During these interviews, students were presented with 6 open statements and graphs of 4 functions and then asked to justify why the statements were true or false for each individual function. The six open statements are presented in Table 1, and the four graphs which were given are shown in Figure 1.

Table 1. Statements given in ETI 1 & ETI 2.

Statements	Statement Type
Statement 1 There exists a real number c in $[^-1, 8.5]$ such that for all real numbers r in $[^-1, 8.5]$, $f(r) \leq f(c)$.	$\exists x_1 \forall x_2$
Statement 2 For all real numbers w in $[^-1, 8.5]$, there exists a real number k in $[^-1, 8.5]$ such that $f(w) \leq f(k)$.	$\forall x_1 \exists x_2$
Statement 3 There exists a real number m such that for all real numbers p in $[^-1, 8.5]$, $f(p) \leq m$.	$\exists y \forall x$
Statement 4 For all real numbers d in $[^-1, 8.5]$, there exists a real number z such that $f(d) \leq z$.	$\forall x \exists y$
Statement 5 There exists a real number t in $[^-1, 8.5]$ such that for all real numbers v, $f(t) \leq v$.	$\exists x \forall y$
Statement 6 For all real numbers j, there exists a real number q in $[^-1, 8.5]$ such that $f(q) \leq j$.	$\forall y \exists x$

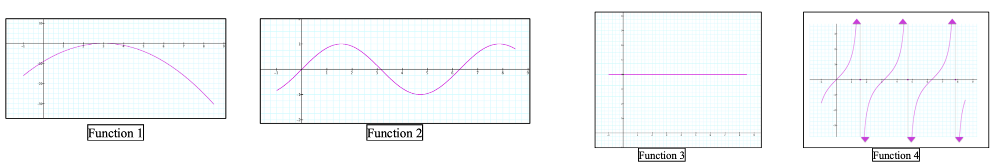


Figure 1. Graphs presented in first two exploratory teaching interviews.

The graphs of functions shown in Figure 1 provided opportunities for me to ask students to explain their thinking about the truth values of the given statements in more detail. Furthermore, these graphs were vetted along with Statements 1-6 in pilot studies and were found to be beneficial tasks for eliciting distinctions in student quantifications. Consider the Extreme Value Theorem (whose conclusion is shown in Statement 1). Since the sinusoidal graph (Function 2) has two values of the input that correspond to the same maximum value, some students in pilot studies stated that Statement 1 is false for Function 2 if they quantified "there exists" as "there exists a unique" (i.e., used MQ2 instead of MQ1).

In the final interview, I presented students with several alternative statements to test how well their quantifications transferred to different mathematical contexts. Additionally, I would

[1] The larger study focused on both students' interpretations and their negations of complex mathematical statements.

often change the quantifier language of these statements and ask interviewees if this language impacted their meanings for the statement. One of these statements is shown below:

Statement A: For all differentiable functions f on $[a, b]$, there exists a c in $[a, b]$ such that $f'(c)=0$. Statement A (Alternative): Suppose f is any differentiable function on $[a, b]$. Then there exists a c in $[a, b]$ such that $f'(c)=0$.

Data analysis. While interviews were being conducted, I made initial models for students' quantifications and interpretations for each statement using Sellers et al.'s (2021) framework and also recorded students' evaluations for each statement across different moments. After each student's data was collected, I completed both a transcription and content log for each student interview. Initial open coding involved determining how to split my data into *relevant moments* (Sellers et al., 2021) where a student explained their interpretation of a statement or explained why a statement was satisfied. I determined that a new moment began whenever a student was presented with a new question or task, the student changed their evaluation of a statement, or if the student provided a new interpretation of a statement.

Once relevant moments were identified, I conducted iterative coding using Sellers et al.'s (2021) framework, as I could, for each student one at time, moment-by-moment, until I had codes that explained student quantifications across all moments for all students. However, a new code emerged in this process of trying to best explain student quantifications. I then re-observed students' words, gestures, and markings on graphs in order to characterize their mental actions that are associated with their quantifications until all the codes I made had explanatory power across all moments for all students in the study.

Finally, I used the evaluations that I coded along with the raw student data in order to determine why students evaluated statements in particular ways. Codes emerged in this phase for meanings that comprised students' interpretations of complex mathematical statements that also explained students' evaluations. For purposes of this paper, I only focus on one code or theme that emerged that related to students' interpretations of complex mathematical statements: students' meanings for quantifier language.

Results

In this study, three of the eight students in my study exhibited different quantifications *consistently* depending upon the statements at hand. Two of these students were in Transition-to-Proof at the time of the study, while one student was in advanced calculus. For purposes of this paper, I focus my attention on one student, Allison, who was currently in Advanced Calculus at the time of the study. I use Allison as a way in which to highlight some of the implications of quantifier language upon mathematical meanings.

The set-wise collection meaning and its connection to "all." The three students mentioned above each claimed that the word "all" led them to think of some variables as sets, and described how they interpreted and evaluated other statements with different quantifier words. Statements 1-6 all have universally-quantified variables and each of those variables are attached to the word "all." When students described these variables, I noticed that they often used additional words such as "all *at once*" to convey that they imagined grouping the values of the universally-quantified variables. In this subsection, I describe how student words for the values of the universally-quantified variable such as "all at once," "whole interval," and "entire set" helped me identify and describe this quantification.

I utilize the following moment with Allison to highlight how the word "for all" impacted her interpretation of Statement 2 ($\forall w(\exists k\, f(w) \leq f(k))$) along with Graph 4 (the modified tangent graph). Before this moment, Allison had determined that Statement 2 was false for the modified tangent function. During this moment, she was in the process of analyzing her evaluation for graphs against other student arguments. I asked Allison to examine several alternative student responses. I referred to an alternative student, Emma, and her argument for Function 4. Below is the argument I presented to Allison:

Alternate Student Argument: Emma said, "If I put w at 0, then $f(w)=0$, and if I put k at 8, $f(k)$ is approximately 1. In this case, the statement is true because $f(w)$ is less than or equal to $f(k)$. However, if I put w at 1.5, $f(w)$ would be approximately 1, and if I put k at 0, then $f(k)$ would be 0. In this case, the statement is false for this graph because $f(w)$ would be greater than $f(k)$. Therefore, I think that Statement 2 is sometimes true for this graph."

I presented Emma's argument for Function 4 in order to examine Allison's quantifications for each variable and why these quantifications led her to a false evaluation of Statement 2 for this function. I designed Emma's argument to represent the use of MQ4 for the variable w, since Emma checked the predicate for values of w individually, and then concluded that the statement is sometimes true. Allison rejected Emma's argument in the moment below based upon her own quantification for w.

Interviewer: [...] Do you think Emma's argument is the same as your argument?
Allison: She [Emma] was still just considering it one piece at a time, not all cumulatively (gestures as shown).

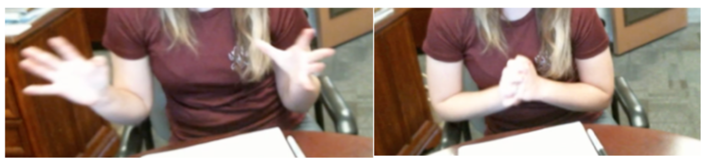


Figure 2. Allison's gestures when discussing w in Statement 2.

Interviewer: When you say all cumulatively, what are you thinking about in your head [...]?
Allison: I think if [...] you put k at 8, then you can't just check the case that w is 0 and say that that's the case. Because the statement is asking about all $f(w)$'s, I kind of feel like you have to like check them all. […] (still explaining Graph 4 from her perspective) Since this [function] is going to infinity, if I choose one [function output] that's higher up, and I say this [new point] is now w, then that's going to be false because you didn't choose a k for all of these numbers [$f(w)$] [...]
Interviewer: So, do you have to think about all of the values of w and all of their output values together at once?
Allison: Yes.
Interviewer: Okay, can you explain that a little bit more?
Allison: Well, you definitely have to think about all of the w's, because it says "for all." And if we're thinking about all the w's, then we're thinking about all their outputs being less than or equal to this one k and $f(k)$.

In determining Allison's meaning for the variable w, I used her descriptions of the statement, the values of w in relation to Function 4, as well as her gestures. First, I noted that Allison used the words "all cumulatively" which suggests that she was mentally grouping the values of w together. Additionally, when I asked if she was considering the values of w and $f(w)$ "all... at once," she confirmed that this is how she was thinking about the values as well. Beyond her description of her own thinking, she also stated that if she changed the values of w, that this would not be an accurate interpretation of the statement, in her view, because then she could not choose a k for *all* of the values of w, and that thinking about all values made her think that all of the values of w together had to be less than a singular $f(k)$ value. Additionally, Allison's gesture in Figure 2 that accompanied her words "all cumulatively," further indicates that she is collecting all the values of w into one set. I coded Allison's meaning for the variable w in this moment as a "set-wise collection" meaning for a quantified variable. (In light of Sellers et al.'s (2021) framework, I extended the framework and will refer to this meaning as MQ5.)

There are two different ways that a student with a normative meaning for "for all… there exists…" ($\forall\exists$) and "there exists… for all…" ($\exists\forall$) statements might map elements of one set of values to another set. First, for a statement of the form $\forall x \exists y$, *each x* may be mapped to a different y. Yet, for a statement of the form $\exists y \forall x$, the same y-value is mapped to *each x*. Yet, for the student who uses a set-wise collection meaning views "For all x, $P(x)$" as indicating that he needs to check the predicate for the *entire set*, X, at *once*, and grouping the set of x's into X before the predicate is checked. For a quantified statement of the form $\forall x \exists y$, the student who uses a set-wise collection meaning for x will group the entirety of X at once *before* mapping and searching for the existence of a y-value. As a result, the student needs a *singular y-value* to satisfy the predicate for every x-value. Thus, for both $\forall\exists$ and $\exists\forall$ statements, the student will refer to an individual value of y that must satisfy the predicate. This set of cognitive steps is shown as a mapping in Figure 3.

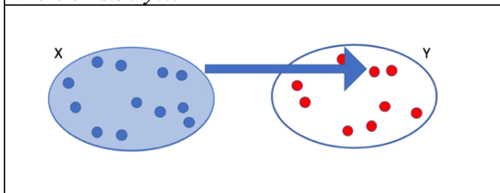
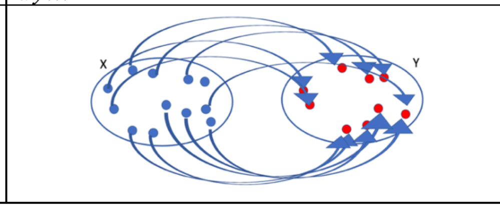
Set-wise Collection Meaning for x in "For all x… there exists a y…"	Conventional Meaning for "For all x… there exists a y…"
	

Figure 3. Illustration for student using set-wise collection meaning for universally- quantified variable.

Six out of the eight students (three Transition-to-Proof students and three Advanced Calculus students) I interviewed had some moments where they used MQ5 for quantified variables. MQ5 was used by these students only for universally-quantified variables, and typically when the quantifier word "all" was given in a statement.

How "any" leads to alternative quantifications. There were some cases in this study where $\forall\exists$ statements using the words "all," "any" or "every" led to different cognitive processes than normative interpretations. In the next example, I present a moment with Allison to explain how students' quantifications for the word "any" can also deviate from convention.

On Day 3, I also posed both Statement A and an alternative version of Statement A, using the word "any" instead of "all" (see methods). While Allison did interpret Statement A as an $\forall\exists$ statement, she did not interpret "for any... there exists..." as an $\forall\exists$ statement. Below, I present Allison's description of how this language change impacted her interpretations of the two different statements in the moment below:

Interviewer: So suppose I change this word here "all" [in Statement A] to the word "any." Would that change the meaning of Statement A?
Allison: Seems like it kind of does.
Interviewer: Ok, how so?
Allison: Because, if you say "for any," I feel like that means like you have all of these choices of differentiable functions, and you pick one of them, and you happen to get lucky and you pick one that satisfies that this condition, that it's true.
Interviewer: Okay, alright, so the word "any"-how does that… how is that different than the word "all?"
Allison: So "for all" I feel like you have to […] go through all of them and make sure they all satisfy this condition. Whereas "for any," you have a choice of which one, which differentiable function you pick.
Interviewer: So, then, how would you evaluate that statement if it had the word "any" there instead of "all?"
Allison: I think it would be true, because when I hear the word "any," I just think of like […] any one you want.

Allison was classified by her use of MQ1 (that at least one value of f satisfies the predicate) for the variable f in the alternative version of Statement A that contains the word "any." I classified her meaning as MQ1 because she said that "any" made her think that she could select *one* function out of all of them that would make the statement *true*. Allison interpreted "any" as "any one you want," and she explained that she wanted one that would make the statement true. This means that for Allison, "any" indicates that you can *choose one* from *an entire set*. As a result, she interpreted the alternative version for Statement A like a "there exists... there exists..." statement, rather than as an ∀∃ statement.

The example with Allison above reveals that the change of the word "all" to the word "any" in an ∀∃ statement should not be considered an automatic fix that will help all students interpret the statement as an ∀∃ statement. Additionally, one should note, changing "all" to "every" is not an automatic fix either, as another student used MQ5 when he was given the word "every."

Conclusion & Discussion

One of the main findings of this study was a new theoretical addition to Sellers et al.'s (2021) framework for student meanings for quantified variables. I call this meaning MQ5, or the set-wise meaning for quantified variables. In Table 2 below, I show how the framework has been extended to include this new meaning.

Table 2. Characteristics of MQ5, the set-wise collection meaning for quantified variables.

Observable Behaviors	Mental Actions
• Identifies a domain of discourse, X. • Explains or illustrates whether or not X, the collection of all x's, satisfies the predicate. (May use gestures such as marking off boundary or use a grouping gesture.) • Uses words such as "whole," "the set of," or "entire" to refer to the collection of values of x in X; Uses words such as "all" or "all at once" to refer to X's satisfaction of the predicate.	1. Identifies a domain of discourse, X. 2. Determines individual values (x_0) that comprise X. 3. Considers the collection of values of x from X as a singular unit. 4. Checks if the predicate is satisfied by the collective set X at once (i.e., checking P(X), not $P(x)$).

In addition to this new meaning for quantified variables, this study also contributes to our knowledge on how language choice in mathematical statements impacts student quantifications. The word "for any" tends to lend itself to having students use MQ1, because they think "any" implies "any one that makes a statement true," whereas "for all" might tend to lead students to the use of MQ5, or a set-wise collection meaning for a quantified variable.

Additionally, this study connects how the language of mathematical statements might relate to students' interpretations of complex mathematical statements holistically. The word "for all" led students to using the set-wise collections meaning for quantification, which then led them to treat ∀∃ statements like ∃∀ statements, similar to Vroom's (2022) finding. The theoretical construct of MQ5 allowed me to explain how students equate ∃∀ and ∀∃ statements in their minds by using set-theoretic interpretations of these meanings. On the other hand, the word "any" led the same student to treat a different ∀∃ statement as an ∃∃ statement.

This study suggests that in teaching and in curriculum development, certain quantifier language may be more advantageous in certain mathematical contexts than others (e.g., in presenting definition of a function, "each" or "every" might be more productive than using "all"). However, a change from one quantifier word to another is not a "cure all" and may bring other unconventional quantifications to light rather than highlighting the intended meaning of the statement for all students.

Acknowledgments

This research was made possible by an institutional research grant.

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