# Student Quantitative Understanding of Integration in Single Variable Calculus

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*Student understanding of integration and key concepts from Single Variable Calculus and the role of infinitesimals was investigated. Analysis was done using a framework which highlights quantitative reasoning. Data showed rich understanding of integrals in multiple representations, with some gaps. Also revealed were connections between those concepts expressed over intervals in Algebra and expressed instantaneously in Calculus, and how conceptions of infinitesimals supported those connections. Implications for instruction are considered.*

*Keywords*: Calculus, quantitative reasoning, infinitesimals, integral

## Introduction & Literature Review

First semester Calculus is a critical course for all STEM majors, a required course containing foundational ideas for future study. Student success has been below desired levels for a long time, and efforts toward improving student success are extensive and ongoing (Bressoud, Mesa & Rasmussen, 2015). There has been much investigation into student understanding of Single Variable Calculus (SVC), exploring conceptions of limit (Oehrtman, 2009; Tall, 1992), derivative (Zandieh, 2000; Park, 2013), integral (Jones, 2015; Sealey, 2014), and the Fundamental Theorem of Calculus (Carlson, Smith & Persson, 2003; Radmehr & Drake, 2017). Some work has been done to connect those conceptions to a student's related prior conceptions (Thompson, 1994; Pustejovsky, 1999; Samuels, 2011). However, none of these studies investigated these questions for students learning Calculus using infinitesimals.

Recently there has been a growing call to revise Calculus instruction away from an Analysis-based approach using limits and toward a more quantitative approach (Augusto-Milner, Jimenez-Rodriguez, 2021). There have been a handful of documented attempts to do so with infinitesimals (Ely, 2021). Some research has theorized the underlying student thinking in such an approach (Ely & Ellis, 2018). None of these studies of infinitesimal-based Calculus instruction has examined how students conceive of important Calculus ideas. These gaps in the literature suggest the following research questions. (1) What is a characterization of student understanding of integration and key Calculus concepts after one semester of Calculus taught with infinitesimals? (2) How are understandings of Algebra concepts in discrete form used to generate understandings of related Calculus concepts in instantaneous form?

## Framework

To analyze student understanding of Calculus, an approach oriented towards quantitative reasoning (Thompson & Carlson, 2017) was used. This mode of reasoning entails "conceptualizing a situation in terms of quantities and relationships among quantities" (Thompson & Carlson, 2017, p425), where a quantity is a measurable attribute combined with a way to measure that attribute. The quantities of SVC and the relationships between them can be described using the ACRA Framework (Samuels, 2022). The framework is summarized here briefly with a delineation of the quantities and some of their relationships. An *amount* is a real value, a *change* is a difference between two amounts, a *rate* is a ratio of two changes, and an *accumulation* is a sum of consecutive changes. One important relationship is that the product of a rate with a change (in the input variable) is a change (in the output variable), the *change equation*. Changes and rates over real intervals are encountered in a typical Algebra course;

changes and rates over arbitrarily small intervals first arise in SVC. A positive infinitesimal is a positive number which is smaller than all positive real numbers. Instead of using limits, one can interpret differentials as infinitesimals to describe the arbitrarily small quantities in Calculus. (For further details, see (Samuels, 2022).) This study was targeted specifically at student conceptions after taking Calculus I. In a previous study (Samuels, 2023), I focused on conceptions of instantaneous rate. In this study, I focused on conceptions of accumulation along with connected ideas and relationships.

*Table 1. Part of the ACRA Framework for Quantities in Calculus (from (Samuels, 2022))*

| **QUANTITIES** | **Description** | **Real** | **Infinitesimal** |
|---|---|---|---|
| **Amount** | A magnitude or extensive quantity | $x, y$ | ε |
| **Change** | A difference between two amounts | $\Delta y = y_2 - y_1$ | $dx$, $dy$ |
| **Rate** | A quotient of two changes (of different quantities) | $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ | $f'(x) = \frac{dy}{dx}$ |
| **Accumulation** | A sum of consecutive changes | $f(b) - f(a) = \Sigma_i \Delta y_i$ | $f(b) - f(a) = \int_{x=a}^{x=b} dy$ |
| **RELATIONSHIPS** | | | |
| **Change Equation** | The product of a rate with a change in one quantity is a change in the other | $\Delta y = m \cdot \Delta x$ | $dy = f'(x)\, dx$ |
| **CONVERSIONS** | An equation of amounts converts to one of infinitesimal changes or rates, and those two convert back and forth | $y = f(x)$ | $\Rightarrow dy = f'(x)dx$ <br> $\Updownarrow$ <br> $\Rightarrow \frac{dy}{dx} = f'(x)$ |

## Methodology

In 2023 Spring the author taught a Calculus I course at a northeastern college designed to use infinitesimals instead of limits. Two months after the end of the course, a clinical semi-structured interview (Hunting, 1997) was conducted with one of the students, who had volunteered. A script was prepared beforehand with questions on change, rate, accumulation, and the relationships between them, in both precalculus and calculus contexts, and in multiple representations (verbal, numerical, graphical, symbolic). Flexibility was allowed for follow-up questions which might be suggested in-the-moment by student responses. The interview was video recorded, transcribed, and coded using the framework.

## Results

Travis [a pseudonym] was first asked "without Calculus, what's an example where we could talk about change?" He explained that "you could have two different amounts, and you basically subtract one from the other to get change". He then described an example involving toys produced, made a table of values, calculated various changes, and drew the graph in Figure 1a. When asked about rate, he expressed it as the ratio $\Delta y/\Delta x$ and calculated it over several different intervals (see Figure 1b).

When asked about accumulation, Travis said that "it's the sum of each individual change in $x$ or $y$ value." For the question, "If a plant grows 3in/week for 2 weeks, then 2in/week for 4 weeks, then 1in/week for 6 weeks, how much did it grow?", his calculations are in Figure 2a. He initially wrote the numerical calculations, and when prompted for "a more abstract notation" he

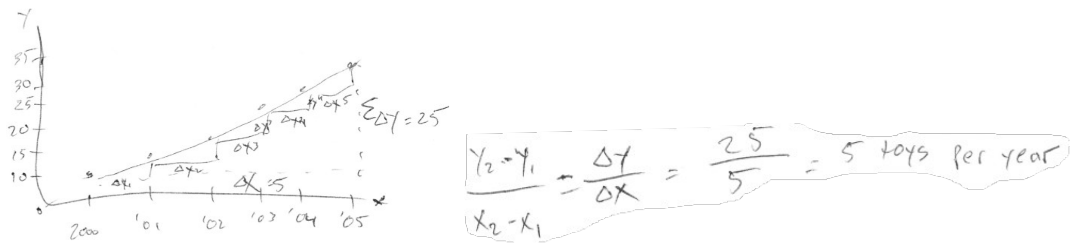


*Figure 1. Travis' example to explain change and rate on a real interval (a) Travis' graph (b) Travis' calculations*

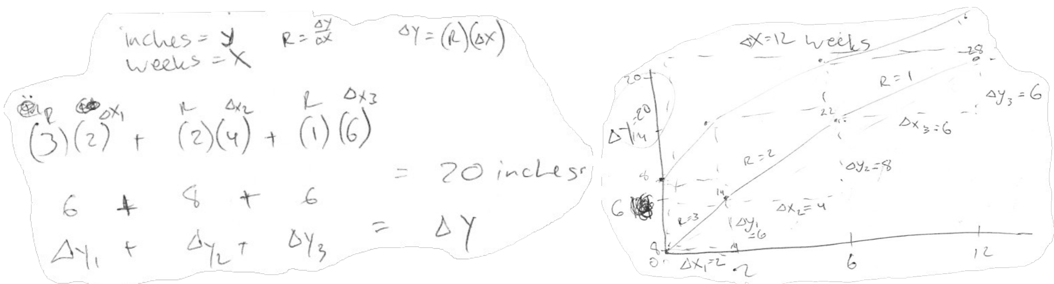


*Figure 2. On a question to calculate an accumulation with a finite sum (a) Travis' calculation (b) Travis' graph*

added more lines, expressing the accumulation as "$R \cdot \Delta x_1 + R \cdot \Delta x_2 + R \cdot \Delta x_3$" ("*R* for rate") and "$\Delta y_1 + \Delta y_2 + \Delta y_3 = \Delta y$".

Travis was asked to make a graphical representation of his calculations, and his work is in Figure 2b. It includes changes in *x* (e.g. $\Delta x_1$=2), changes in *y* (e.g. $\Delta y_1$=6), rates (e.g. *R*=3), and total changes $\Delta x$=12 and $\Delta y$=20.

Next, questions involving Calculus were asked. When asked "What is a situation where you would need calculus to talk about the slope or the rate?", Travis replied "You need calculus when you do not have a constant rate on a line, and you want to find the instantaneous rate at a certain point." He elaborated by creating the example $f(x) = x^2$ and then calculating $dy/dx = 2x$, which at the point (4,16) was 8 (see Figure 3b). In the following excerpt, he describes his reasoning.

*Interviewer*: Can you indicate on this graph [Figure 3a] what a *dx* is?
*Travis*: Sure. You would have to zoom in really closely on the point [draws a circle], and then you would have a little *dy* and a little *dx*. [draws and labels two line segments]
*Int*: And what do you mean by little?
*T*: Infinitely small. ...it's a little hard for me to describe it, but zoomed all the way in on that point, you get a line. The slope of that zoomed-in line is your infinitesimal rate.

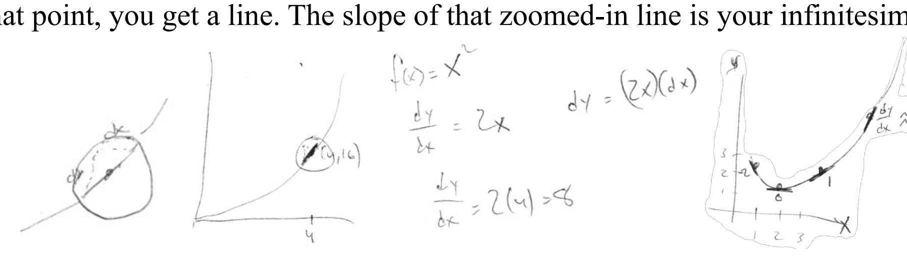


*Figure 3.(a) Travis drew dx and dy (b) Travis created a function, graphed it, calculated the derivative and dy (c) Travis drew the graph, marked several points, and for each drew a short tangent line and a slope estimate*

Given the graph of a function, Travis was asked to draw tangent lines and estimate the derivative at several points. He did this correctly, as shown in Figure 3c. When asked to solve for *dy* in his example, he wrote "$dy = 2x\ dx$". When asked what it means, he said "It means that the infinitesimal change of *y* equals the infinitesimal change in *x* multiplied by the rate."

When asked about infinitesimals, Travis said "you only have a point to work with, so you need to zoom in so far on that point in order to get a line... And the infinitesimal basically gives you a way to use real numbers to assign values to the changes going on at that infinitely small point." He said it is not possible to write down an infinitesimal number, so we use the symbol ε.

Travis was given an application in which the height of a ball was $h = 27t - 10t^2$. When asked for the rate, he calculated $dh/dt$ = 27-20$t$; when asked for the units he said "meters per second". When asked to compare $\Delta h$ and $dh$, he said "because you're dealing with $dh$ and $dt$, these are infinitely... small changes in the ball's height, because you're only doing it in relation to an infinitely small change in the time. If it was going to be $\Delta h$ and $\Delta t$, it would be a bigger change. So your $\Delta t$ may be 2 seconds, in which case your $\Delta h$ would be a bigger number."

When asked to compare $dh$ versus $dh/dt$, he said that "$dh/dt$ is a rate, and it is...the rate at which the height of the ball is changing in relation to time... And $dh$ by itself is just the change in height of the ball, not the rate at which that height is changing."

When asked about integration, Travis described it as "calculating the sum [of changes] between two points". As to types of accumulation, when asked "is there a notation for doing" the sum in a finite accumulation, Travis responded "the sigma thing". When asked to compare with the "Calculus version of the same thing", Travis responded "this is going to be my integral".

When given a slope field and asked about its meaning, he said "these are all representations of, I guess, instantaneous rates of whatever the function is of these graphs" and each dash "tells you the direction that the function is going to be going at that point."

Travis was asked an application question. "A rope has a certain amount of mass $m$ over its length $x$. Suppose it has $2x$ kg/ft at the $x$-foot mark. How much mass does it have between the 1-foot mark and the 4-foot mark?" Travis' correct calculations are in Figure 4a. He was also asked to sketch a graphical representation of the solution, and his inscriptions are in Figure 4b. It includes a slope field, a solution curve, with $\Delta x$=3ft and $\Delta m$=15kg marked. (He noted that "obviously my scale is all messed up".) He explained his work in the following excerpt.

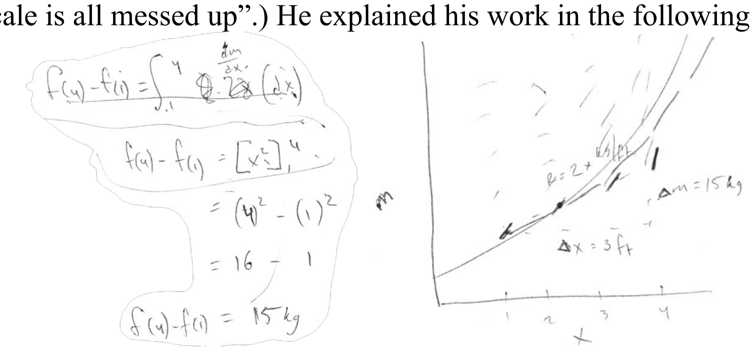


*Figure 4. For an application to find mass given density (a) Travis' calculation (b) Travis' graph*

*Int*: The first expression you wrote down, where did that come from?

*T*: You're asking from the 1 foot to the 4 foot mark. So those are my limits...

*Int*: ...in the original expression, it's the integral from 1 to 4, $2x\ dx$. So tell me how that's going to give you $f(4) - f(1)$ … Tell me what that means and how that's the thing that you need.

*T*: This [pointing to $f(4) - f(1)$] is my $\Delta y$, for lack of a better phrase. And then this [pointing to the right side] is my rate times my $\Delta x$...I know that by multiplying my rate and my $dx$ will give me my $dy$... so the $2x$ times the $dx$ that gives you a $dy$.

*Int*: ...what about the actual squiggle, the actual integral mark itself? Is that telling you something specific or is just something that you write down?

*T*: The squiggle is like the letter S stands for sum.
*Int*: And what is it that you're adding up? If it's a sum, what are you adding up?
*T*: The values of the mass between 1ft and 4ft. Yeah, the value of mass at 1ft plus the value of mass at 2ft, or I guess at every infinitesimal increment between 1 foot and 4 foot to give you the amount that it has between those two points.

While explaining his calculation, Travis noted that he was expecting an $x$ to appear in the answer. He explained his confusion about it, eventually sorting it out and referencing the FTC.

> I'm thinking I'm... trying to figure out where this $x$ mark on the rope is, but that's not what you're asking… There would be other times where you would be asking, how much mass does it have between the 1ft mark and the $x$ foot mark? … I would even have been happier having the limitations from 1 to $x$ and be solving for $f(x) - f(1)$ … This is very simple.

In the excerpt below, Travis described some infinitesimals and their quantitative characteristics.

*Int*: What does *dm* represent?
*T*: An infinitely small change in the mass of the rope.
*Int*: Okay. And in the course of doing this calculation or this problem, how many *dm*'s are you going to encounter?
*T*: Infinitely many.
*Int*: Okay. And *dm/dx*, what does that represent?
*T*: That would be my $2x$. That would be my, that's my $m'(x)$, my instantaneous rate.

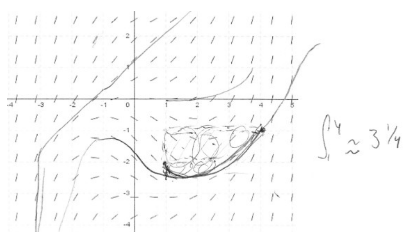


*Figure 5. Travis represents the integral on a slope field.*

When asked to use a slope field to find the integral from $x$=1 to $x$=4, Travis said "If you have an endpoint at four and endpoint of one, you have this little area here of your change in $y$ from one to four and your change in $x$. I would shade in this and say that's your integral from one to four… your $\Delta y$ accumulation would be one and a half-ish and your $\Delta x$ would be three… Basically, you're going to find this area [counting boxes] one, two, three, and like a quarter." His inscriptions are in Figure 5.

In the excerpt below, he discussed two different graphical methods of representing integrals.

> When I was taking the class ... I was going to tutoring, and [he] explained [integration] to me using area…I was like, okay ... but slope field ... at the time I don't remember even linking it to integrals, which obviously now I see how it does. But all I remember was like, wow, this is easy. I just draw a line going along with the other lines... this is simple.

## Discussion

Travis demonstrated a detailed understanding of the quantities of Calculus. He gave clear explanations of the concepts of change, rate, and accumulation, and made specific connections between them. For change, he represented a real change numerically as a difference between two real numbers and graphically as a line segment; he represented an infinitesimal change both symbolically (using $dx$) and graphically (using zooming). For rate, he represented discrete rate numerically and symbolically as a ratio of two changes and graphically as the slope of a straight line. For an infinitesimal rate, he calculated one using formulas, represented it graphically as a short tangent line (both on the original graph and after an infinite zoom, and subsequently as a slope field), and described it an instantaneous rate, using both prime and differential notation. (Although he did not explicitly refer to an infinitesimal rate as the ratio of infinitesimal changes, he frequently manipulated it as though it were.)

Travis was able to perform all 3 conversions, from an amount equation to an infinitesimal change or rate equation, and between the last two. From an equation with "$y$=", in addition to calculating the derivative, he wrote down the infinitesimal change equation with "$dy$=". Further, he was able to assign a quantitative meaning to each term - the infinitesimal change in $x$, the infinitesimal change in $y$, the infinitesimal rate - and interpret the equation as a relationship between those quantities, both in an abstract scenario and an application where he assigned correct units to each. Also, he distinguished between infinitesimal rate and change, in meaning and use. He converted between the instantaneous change and rate equations by multiplying or dividing by $dx$. In a standard Calculus course, students are told both that one cannot manipulate in that way and individual differentials have no meaning, as well as to manipulate that way and use isolated differentials in the procedure of integration by u-substitution. The inherent contradiction has been noted before (Ely, 2021). After Calculus instruction with infinitesimals, Travis had a coherent concept image (Tall & Vinner, 1981) of these equations, in meaning and in symbolic manipulation.

Travis was able to describe accumulation using the quantities of change and rate. In the finite case, he stated that it was the finite sum of changes; given a piecewise constant rate, he was able to calculate the accumulation using arithmetic, and graph it as a piecewise linear graph. He also stated that the accumulation was a sum of infinitely many infinitesimal changes; given a formula for instantaneous rate he calculated the accumulation with an integral, he graphed it as a solution curve in a slope field. He also (after some confusion) described the calculation both for a constant upper boundary to produce a constant answer and a variable upper boundary to produce a function answer (the FTC).

Travis expressed accumulation as a finite sum of changes in the output variable as well as a sum of rate times input variable change terms ($f(b) - f(a) = \Sigma_i \, \Delta y_i = \Sigma_i \, r_i \cdot \Delta x_i$). For integration, he made a parallel infinitesimal statement, $f(b) - f(a) = \int f'(x)\, dx$. Although he did not explicitly write the formula $\int dy$, in one application he did describe the integral which calculated the total mass as a sum of infinitesimal masses, and separately described how $dy$ can be replaced with $f'(x)\, dx$. For integration, previous research has shown that applications are very difficult for students (Wagner, 2018; Jones, 2015), and that quantitative understanding is rare (Jones, 2015; Fisher & Samuels, 2016). Similar quantitative meanings for integration have been discussed in previous studies (Oehrtman & Simmons, 2023; Jones, 2015), but instruction relying on limits results in $dx$ having no rigorous definition.

Travis made strong connections between the real and infinitesimal versions of the same concepts. For change, he indicated $\Delta x$ as a horizontal segment on the original graph, and he indicated $dx$ as a horizontal segment on the graph visible after you "zoom in really closely… [it's] infinitely small". In the ball problem, he referred to $dh$ and $dt$ as "infinitely small changes" whereas "$\Delta h$ and $\Delta t$, it would be a bigger change". For rate, Travis drew (piecewise) lines for (piecewise) constant slopes, labeling slope values. For instantaneous rate, he described "zooming in" at a point to reveal a straight graph; he drew that straight tangent line on both the original graph and a zoomed-in graph, also marking the slope value at a point. For accumulation, he wrote the formula as a finite sum with Σ, and he used ∫ as an infinite sum of infinitesimals, and integration would be needed when the rate is non-constant. He noted that there would be "infinitely many" *dm*'s and *dx*'s in calculating the integral for mass. He made a direct comparison between $\Delta y$ and $dy$ in the context of rate, change, and accumulation. In various contexts, Travis made statements about real quantities and incorporated infinities and infinitesimals to express corresponding statements. Thus he exhibited *transfinite thinking* (Samuels, 2023) underpinning his Calculus conceptions. Learner connections between finite and infinite scenarios have been investigated previously (Oehrtman, 2009; Mamolo & Zazkis, 2008), but only recently was this term rigorously defined, or were the connections examined for their contributions to productive conceptions.

One initial point of confusion for Travis concerned whether or not the solution to an integration application should contain a variable. Previous research into the Fundamental Theorem of Calculus (FTC) has shown the extreme difficulty students have in recognizing the upper bound as the variable (Thompson, 1994). In this investigation, Travis eventually remembered the two relevant types of question, a fixed upper bound or a variable upper bound, even commenting that he would be "happier" with the variable problem, exhibiting knowledge of the FTC and the functional nature of its formula.

Another point of confusion was the graphical representation of the integral. Working from a slope field, Travis drew a solution curve, then shaded an area and stated that was the result of the integral. He conflated the two graphical representations, the solution curve in a slope field, and the area under the integrand function. This is a conception not previously documented, partly due to the dearth of research on Calculus instruction with infinitesimals. Travis discussed one possible cause, working with a tutor whose instruction did not align with the approach of the class. Previous research has shown that conflicting instruction can cause productive notions of infinitesimals to erode (Simmons et al., 2022). Having two separate graphical representations for the same situation or calculation is inherently confusing (the integrand plays different roles). One implication for teaching is the need to emphasize this distinction as much as possible. The optimal pedagogy is a topic for future inquiry. Other directions for future research include investigating quantitative conceptions of the rate equation, and the FTC.

It is important to point out that data in this study was drawn from a single student. I cannot claim that any observations generalize to the entire population. Rather, the analysis sheds light on how a student might form conceptions about Calculus and integration using infinitesimals. Due to the relative novelty of the context, this signifies a contribution to the literature.

## Conclusion

In this study, I examined an understudied area of student understanding in Single Variable Calculus, conceptions of integration after one semester of Calculus I taught using infinitesimals instead of limits. The data showed rich connected student conceptions of integration, supported by conceptions of change and rate, in multiple representations. These conceptions were expressed both for real quantities and infinitesimal quantities, connected by the student via transfinite thinking.

Calculus instruction using infinitesimals is not typical practice. There is a growing body of evidence that it is a productive approach for students to make quantitative meaning in Calculus, and this study is a contribution in that direction. It also demonstrated how student knowledge of the quantities and relationships within Calculus, and their connection to prior knowledge, can be exhibited by students, and productively analyzed using the ACRA Framework.

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