Student Interpretations of Conditional Statements in Probability: The Cases of Byron and Sophie

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Students often have to interpret conditional statements in mathematics classes without prior training in logic. In a previous study, we investigated how students interpreted variants of the conditional statement: If two random variables are independent, their covariance is 0. The purpose of the current study is to explore surprising results and apparent inconsistencies in student interpretations using clinical interviews after a lecture introducing the conditional statement. In this paper we contrast the experiences of two students, Byron and Sophie. Preliminary results suggest that Byron relies both on his logic training and his domain knowledge to solve problems about independence and covariance. Sophie's incorrect domain knowledge led to her reversing the implication. Once this was resolved, she was not confident in assigning truth values to statement variants. We discuss teaching implications that arise from issues of both mathematical efficiency and accuracy in presenting this theorem.

Keywords: Probability, Conditional Statements, Logic, Covariance, Independence

Undergraduate math courses typically present multiple definitions and theorems in the form of conditional statements. We know students often do not interpret these statements in the mathematically normative manner, and learning how to interpret conditional statements is particularly challenging due to epistemic issues related to set-theoretic components of conditional statements (Case, 2015; Dawkins & Cook, 2017; Durst & Kaschner, 2020; Sellers, 2020). In a previous study (Ryals, et al., 2023), we analyzed Probability students' written work to determine how they interpreted and used the following theorem: "If X and Y are independent random variables, then $\mathrm{Cov}(X, Y) = 0$" (Casella & Berger, 2002, p. 171). We also considered whether students' prior logic training, or lack thereof, would impact their interpretations. Approximately 1/3 of the participants incorrectly used covariance being 0 as justification for two random variables being independent, and 10% of students failed to use covariance being nonzero as justification for variables being dependent. The results did not significantly differ for students who had and had not previously received logic training in a prior course.

In Spring 2023, we began conducting clinical interviews (Clement, 2000) with students in the same course in an effort to investigate results from our prior study. Specifically, we address:

1. How do students view the relationship between covariance and independence?

2. How does their view of this relationship influence how they use the value of covariance to determine and justify whether two variables are independent?

In this report, we present preliminary findings from interviews with two students, Byron and Sophie. We discuss and compare their conceptions of covariance and independence and the relationship between the two as well as how and why they used covariance values to determine independence of two random variables.

Review of Literature

For multiple reasons, students often do not interpret conditional statements in conventional ways and the same student can interpret logically equivalent statements differently in different contexts (Dawkins & Cook, 2017; Durand-Guerrier, 2003). Students often interpret conditional statements as biconditionals (Hoyles & Küchemann, 2002; O'Brien et al., 1971; Wason, 1968) which is often *intended* in common language (Epp, 2003). Moreover, a statement indicating P implies Q, devoid of quantified variables, is an open sentence, which could be considered true for one case and false for another (Durand-Guerrier, 2003). Yet, mathematicians typically read these statements as generalized conditional statements. This means we interpret our theorem as saying "*For all pairs* of random variables, if X, Y are independent, then $\mathrm{Cov}[X, Y] = 0$." However, the implicit "for all" quantifier is not typically stated explicitly, which understandably leads to non-normative evaluations for some students (Durst and Kaschner, 2020).

Efforts have been made to improve teaching students how to interpret conditional statements appropriately within the mathematics register (Dawkins & Norton, 2022; Dawkins, et al., 2023). Evidence suggests students have an easier time interpreting verbal and mathematical statements than abstract or symbolic ones. Consequently, one approach has been to use colloquial statements whose inverse, converse, and contrapositive are obviously true or false to students through context or familiarity as an introduction to the topic (Case, 2015; Epp, 2003, Stylianides et al., 2004). However, training students to generally place a label of "true" or "false" on specific variants such as the converse has not proven to transfer to using those variants appropriately in novel situations in mathematics (Sellers et al., 2017; Attridge et. al, 2016; Cheng et. al, 1986; Inglis & Simpson, 2008). This implies that many students have not fully abstracted logic rules with conditional statements across different mathematical domains. Thus, there is a need to study how students interpret conditional statements in different mathematical contexts and determine how mathematical content of a particular statement may impact students' logical inference.

Methods

Data Collection

This study was conducted in Spring 2023 in a semester-long Applied Probability course designed for engineering and computer science majors. The first author taught the lesson introducing the theorem to four sections of the course. After presenting the theorem, students were asked to individually determine whether each of the three variants listed below was true or false in a pre-class quiz. In this paper, we subsequently refer to these statements as the inverse, converse, and contrapositive, respectively, though we did not use this terminology with students.

1. *Theorem: If X and Y are independent random variables, then $Cov(X, Y) = 0$.*
2. *Inverse: If X and Y are not independent, then Cov (X,Y) ≠ 0.*
3. *Converse: If Cov (X,Y) = 0, then X and Y are independent.*
4. *Contrapositive: If Cov (X,Y) ≠ 0, then X and Y are not independent.*

After the quiz, the instructor clarified the fourth statement was true and the second and third statements were false and explained it is possible for two random variables to have a nonlinear relationship and have a covariance of 0. The instructor then worked through two examples related to the theorem. Each required first determining whether variables were independent and then finding the value of covariance. In one example, the variables were independent, so the theorem could be invoked to conclude the covariance was 0. In the other, the variables were not independent, so the covariance had to be calculated. These examples were chosen to demonstrate the theorem's utility in specific situations and limitations in others.

Following a brief discussion, students individually solved two similar problems. In each problem, they were given a joint probability mass function (Figure 1) and first asked to compute covariance and then determine whether X and Y were independent. In the first example, covariance was 0 so independence had to be determined using the definition of independence. In the second example, the covariance was not 0 so dependence was guaranteed. We will subsequently refer to these examples as the Converse problem and the Contrapositive problem, respectively.

Y \ **X**	**-1**	**0**	**1**	$P_Y(y)$
-1	0.2	0	0.2	0.4
0	0	0.2	0	0.2
1	0.2	0	0.2	0.4
$P_X(x)$	0.4	0.2	0.4	

Y \ **X**	**0**	**1**	**2**	$P_Y(y)$
0	0.1	0.2	0.2	0.5
2	0.1	0.25	0.15	0.5
$P_X(x)$	0.2	0.45	0.35	

Figure 1. Probability Distributions used in the Converse and Contrapositive Problems

We invited all students who had either used the converse on the Converse problem or neglected to use the contrapositive on the Contrapositive problem to participate in a 30-60 minute semi-structured interview. Three students agreed to be interviewed one week after instruction and prior to any formal assessments on the topic. The first author was the primary interviewer and the second author prompted the first with additional questions. We used a prepared protocol but allowed for both follow-up questions and probes (Rubin & Rubin, 2005). During the interview, students were asked to provide definitions of covariance and independence and justify their in-class work. We then presented several colloquial statements and asked the students to provide truth values for those statements and their variants to check for consistency or inconsistencies in logic across different mathematical domains.

Analytic Framework and Analysis

During the interviews and throughout analysis we attempted to construct second-order models of students' thinking about their domain knowledge and logic; "those the observer constructs of the subject's knowledge in order to explain their observations or experience of the [student's statements] and activities" (Steffe et al., 1983, p xvi). To achieve this goal, we asked multiple follow-up questions during the interview and the authors discussed multiple possible interpretations of student claims to come to consensus.

We developed several categories of codes related to specific interview questions. Our first group of codes described the participant's conceptions of independence and covariance. We paid particular attention to whether these were formalized or intuitive notions, whether they were normative, and whether linearity was part of the covariance description. We asked them to describe any mental images that they may have had in relation to covariance, in particular. Our second group of codes addressed their beliefs about the relationship between independence and covariance. Finally, a third group of codes attended primarily to responses to questions involving everyday statements to describe the participant's approach to determining truth values of conditional statements in various contexts. In the subsequent section, we present results for two participants, Byron and Sophie, which includes their conceptions of independence and covariance and the relationship between them as well as their level of comfort with logical equivalence. We then draw comparisons between the two and discuss implications for practice.

Results

Byron was invited to participate in the interviews because he did not use the fact that covariance was nonzero to conclude variables were dependent on the end-of-class exercise.

Instead, Byron correctly applied the definition of independence. Byron's approach was particularly interesting since he had identified the contrapositive as true on the pre-class quiz and he had previously taken Discrete Math.

The interview revealed that Byron held normative and robust mental images related to both independence and covariance. He provided both the formal definition of independence as well as an explanation, saying "I think that means the observed value of one of them will not affect what you'll observe for the other." Byron stated that covariance was a measure of a linear relationship and stated that variables could have a relationship that was not linear. He provided an example, saying "…if all of your data fell on a circle, [the two variables] would have a relationship. But I think…the covariance would come out to exactly 0 if it was exactly on a circle." Additionally, Byron was clear about the claims and limits of the theorem. He stated that independence guarantees a covariance of 0 but a covariance of 0 does not guarantee independence.

Byron had not made use of the theorem on the end-of-class Contrapositive problem, which seems surprising in light of the evidence above. However, when he was asked if there was an alternative way he could have justified X and Y being dependent, he immediately said, "Oh yeah, I do remember now. If two random variables are independent, their covariance is 0, is equivalent to saying if their covariance is not 0, they are not independent. So I guess you can tell from [the fact the covariance is not 0] that they're also not independent."

First, we hypothesize that Byron may have been influenced to use the definition of independence on the Contrapositive problem, rather than invoking the theorem, by the examples worked by the instructor just moments before. In the interview, when asked about these two approaches, Byron said they are equally valid, and that invoking the theorem would make more sense in the case where a nonzero covariance has already been established. Second, we note that Byron introduced the word "equivalent" to explain the relationship between the statement and its contrapositive. He made the same argument to justify his choice of truth value for a variant of a colloquial statement, this time introducing the word "contrapositive," which we had not used in the interview. With this example, he used logical equivalence of the contrapositive as his primary justification for his provided truth value.

Sophie, who had not previously taken Discrete Math, was invited for an interview because she stated that X and Y must be independent because $\mathrm{E}[XY] = \mathrm{E}[X]E[Y]$ on the Converse problem. We coded this response as Applying the Converse. Consistently, Sophie had classified the converse statement as true on the pre-class quiz.

Sophie demonstrated a conceptual understanding of independence by comparing selection with and without replacement. However, Sophie's concept of covariance was influenced and limited by a graphic presented in lecture. When asked what it means to say $\mathrm{Cov}[X, Y] = 0$, she replied, "it means that X and Y are not correlated, so the graph would be more like no pattern, a bunch of scattered dots everywhere." A graphical depiction of dependent variables with a covariance of 0 had not been presented. Additionally, when asked what it meant for two random variables to be independent, she responded, "When the expected value of both of them equals the expected value of each of them multiplied." That is, Sophie believed that two variables are independent *if and only if* $E[XY] = E[X]E[Y]$. This was further clarified when we presented Sophie with a situation where $\mathrm{Cov}[X, Y] = 0$ and she concluded X and Y must be independent. Sophie was not intentionally applying the converse. Rather, Sophie had believed she was checking independence *directly using the definition*. Accepting Sophie's definition of independence would also mean believing the theorem is a *biconditional* statement. Yet, Sophie was unsure about this claim because of what she remembered from class.

Interviewer: What is [covariance] comes out to 0; what will that tell you, if anything?
Participant: I think what I wanna say is that they're independent, but (*pause*)
Interviewer: How confident would you be on a scale of 1-10?
Participant: Probably like 5. I don't know. I remember you saying something in class, like there's a caveat.

At this point Sophie believed a covariance of 0 guarantees independence because of her definition of independence, but she did not believe independence guarantees a covariance of 0 because she remembered from lecture there was a "caveat." Sophie was then prompted to review the definition of independence as well as the theorem written in her notes. She quickly revised her claim about the relationship between independence and covariance.

> "Now I'm thinking if they're independent based on that equation, then covariance should equal 0, but then if they're not independent, covariance can still equal 0. But they're not related inextricably. It's like a weird relationship."

At this point Sophie believed independence guaranteed a covariance of 0 but not the reverse. However, when prompted, she could not produce an example of two dependent variables with a covariance of 0. Moreover, she was still hesitant to provide truth values of the statement variants. When Sophie was asked about her pre-class quiz response of false to the contrapositive, she first claimed that response was incorrect, but then hesitated and ultimately did not provide a definitive answer. She responded similarly to a prompt about the contrapositive of one of the everyday statements later in the interview and relied primarily on context rather than logical equivalence.

Discussion

Byron has two distinct advantages over Sophie. First, since his concept of covariance involves linearity, he can conceive of dependent pairs of random variables with a covariance of 0. He can reference his domain knowledge to determine truth values of variants of the theorem. Secondly, he can avoid referencing his domain knowledge when expedient because he is assured of which statements are logically equivalent. Our analysis of Byron suggests that Probability students may rely on their logic training to quickly assess the truth value of variants of conditional statements, regardless of whether they are focused on the mathematical content in the predicate of a conditional statement. On the other hand, a student who avoids using the contrapositive could simply be relying on their knowledge of the mathematics of the statement.

Sophie demonstrates how gaps in domain knowledge, and specifically lack of counterexamples, can prevent students from determining which conditions imply others and lead them to rely on memorization. The power of examples in justification has been noted already in the literature (Zazkis, et al., 2008). We strongly suspect that instructors including a graphical depiction of dependent variables with a covariance of 0 when introducing covariance would have established for Sophie that the theorem's converse was false. Even so, without a strong notion of logical equivalence, truth values of the inverse and contrapositive would still not be obvious. We do not know at this point whether prior logic training gives students an advantage in Probability courses, especially in light of our previous study (Ryals, et al., 2023). However, we certainly value the flexibility Byron had in using both his domain knowledge and logical inference to answer questions confidently. We need to conduct further interviews or larger-scale surveys with more students who have taken Discrete Math and those who have not to make appropriate conclusions regarding logic training. Also, absent from our participants thus far were students who stated the converse was false on the pre-class quiz but then used it during the exercises. We plan to repeat the in-class instruction and invite more interview participants.

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