Affordances of Semantic and Syntactic Proof Approaches for Isomorphism and Homomorphism

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Group isomorphism and homomorphism are core concepts in abstract algebra, but limited work has directly examined student conceptions of homomorphism or how students approach finding particular mappings. Based on interviews with two students, we contrast one student who used predominantly syntactic proof approaches for homomorphism with one who used semantic approaches, noting that both experienced success in finding homomorphisms in most cases.

Keywords: isomorphism, homomorphism, semantic, syntactic

Introduction

Students' understandings of multiple areas of abstract algebra content have received attention, including binary operation (e.g., Melhuish & Fagan, 2018; Melhuish, Ellis, & Hicks, 2020), inverses (e.g., Serbin, 2023; Wasserman, 2017), and functions (e.g., Melhuish, Lew, et al., 2020; Uscanga & Cook, 2022). We build on the attention to functions, in particular the focus on student understanding of isomorphism (e.g., Weber & Alcock, 2004; Melhuish, 2018) and homomorphism (e.g., Melhuish, Lew, et al., 2020). Specifically, we examine students' reasoning when creating isomorphisms and homomorphisms from a proof production perspective. We believe this work holds two contributions to the literature. First, this paper extends our currently limited knowledge of how students understand homomorphism and solve problems related to homomorphism. Second, this paper provides insight into the relationship between students' success while problem solving and their proof approaches.

Literature Review

Isomorphism was one of the first abstract algebra topics studied in the undergraduate mathematics education community (e.g., Dubinsky et al., 1994; Leron et al., 1995). This early work characterized properties students attended to when assessing whether groups may be isomorphic, such as checking cardinality (Dubinsky et al., 1994), determining alignment between the orders of elements, and determining whether groups were abelian (Leron et al., 1995).

Other work, whose primary aim was examinations of proof approaches, contrasted undergraduates' and doctoral students' (Weber, 2001; Weber & Alcock, 2004) or undergraduates' and algebraists' (Weber & Alcock, 2004) approaches to determining whether groups were isomorphic. Weber (2001) also examined approaches to proofs focused on homomorphism. Of note, doctoral students and algebraists relied more on semantic reasoning, wherein their knowledge of properties of groups and holistic understanding of theorems were key aspects of their reasoning and seemed to be important aspects in graduate students' success. In contrast, undergraduates tended to reason syntactically, wherein they largely focused on symbol pushing with the formal definition, and experienced more struggles in proving more difficult statements. Moreover, semantic reasoning was linked to Skemp's (1976) relational understanding construct, which emphasizes understanding both how and why notions work as well as what to do to solve a problem, whereas syntactic reasoning was linked to instrumental understanding, which only requires understanding what to do (i.e., procedural knowledge). More recently, in Melhuish's (2018) replication study, undergraduates at times successfully used properties to reason about whether groups were isomorphic but seemed to check these properties procedurally, suggesting the need for further research on connections between semantic reasoning, relational understanding, syntactic reasoning, and instrumental understanding.

However, explicit discussion of students' understandings of homomorphism has been more limited (e.g., Weber, 2001). Moreover, research on approaches to homomorphism have focused on students' and instructors' language (e.g., Melhuish, Lew, et al., 2020; Rupnow, 2021; Rupnow & Randazzo, 2022), rather than centering their approaches to proof, suggesting a need for a concentrated look at proof strategies with homomorphism.

Methods and Theoretical Perspective

This paper is grounded in a wider study examining instruction in introductory (junior level) abstract algebra at a land-grant university in the Mid-Atlantic United States. Here we focus on two participants, Bryce and Blake, who were in a section taught with a mixture of lecture and activity days. The students' names are gender-neutral pseudonyms, and they/them pronouns are used throughout. Participants each engaged in a semi-structured interview (Fylan, 2005) lasting roughly one hour. The interview occurred after instruction and an exam on group isomorphism and homomorphism had been completed. Interviews were audio and video recorded and any written work was collected. The interview questions analyzed here focused on finding mappings between specific groups: $\mathbb{Z}_3$ and $\mathbb{Z}_6$, $\mathbb{Z}$ and $2\mathbb{Z}$, and two groups given in Cayley tables (one representing $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the other $\mathbb{Z}_4$). If students were able to determine one mapping, they were often asked to determine if another isomorphism or homomorphism could be formed.

The interviews and videos were transcribed and coded in alignment with the definitions of semantic and syntactic proof production given in Weber and Alcock (2004). Semantic proof production is characterized by the use of "instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences" that are drawn whereas syntactic proof production is "written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way" or could be viewed as symbol-pushing (Weber & Alcock, 2004, p. 210). For instance, an isomorphism proof in which a student uses properties and theorems, especially by drawing on their knowledge of the groups involved, would be a semantic proof, whereas a proof in which a student directly uses the formal definition (i.e., for group isomorphism, show bijectivity and the homomorphism property) would be a syntactic proof. From this analysis, we address the following research question: What patterns exist between proof production approaches and success in creating isomorphisms and homomorphisms?

Results

We present vignettes of two students who were generally successful at the tasks, but one approached homomorphism tasks syntactically (Bryce) and the other semantically (Blake).

Bryce

$\mathbb{Z}_3$ and $\mathbb{Z}_6$. Bryce quickly said these groups were not isomorphic because they are different sizes: "Okay, so can't be an isomorphism because this has order three and this is order six, so elements in $\mathbb{Z}_3$ would have to map to more than one element in $\mathbb{Z}_6$, so it's not one-to-one." While Bryce attempted to explain this property, we note their explanation seems to conflate function and one-to-one definitions and does not focus on the groups' structures.

Regarding the homomorphism task, Bryce realized that nontrivial homomorphisms should exist because three divides six. They were initially unsuccessful in finding homomorphisms when trying to map individual elements. However, they found success when they created formulas to describe maps rather than mapping elements, beginning with the map $f([x]_3) = [2x]_6$ from $\mathbb{Z}_3$ to $\mathbb{Z}_6$. They started checking the homomorphism property element-by-element, but then realized that they could do this process generally (see Figure 1). Later, they also generalized their map to include "any multiple of 2… it could be like 2, 4, 6, any of those."

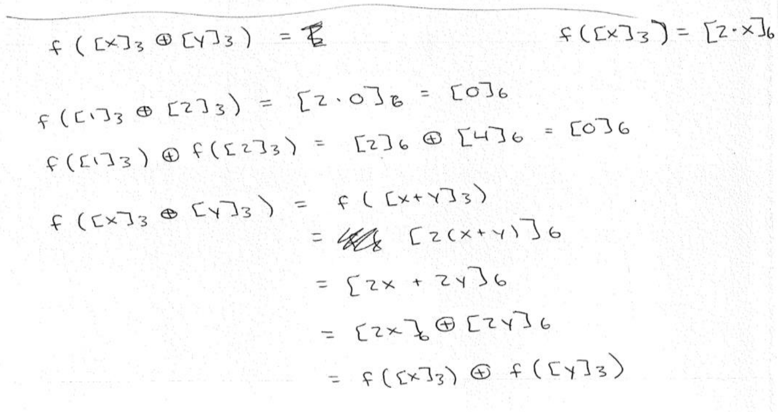

Figure 1. Bryce's Homomorphism from $\mathbb{Z}_3$ to $\mathbb{Z}_6$.

Going from $\mathbb{Z}_6$ to $\mathbb{Z}_3$, Bryce was successful in finding all homomorphisms. In particular, they mentioned the identity map, the map $f([x]_6) = [2x]_3$ ("2 times x… or any multiple of that") and the trivial homomorphism, which they stumbled upon by considering odd multiples.

B: I don't know if an odd number would work too. So if you were just multiplying by 3 instead… I guess if you multiply by 3 here, that would always work cuz that would always go to 0 mod 3, and any multiple of 3 would reduce down to 0 mod 3. So any multiple of 3 would work…

I: So you're mapping everything to 0.

B: Mmhmm…that would be a group homomorphism. Because if you had 1 mod 6 and 4 mod 6, they were both being mapped to 0. Then 0 mod 3 plus 0 mod 3 is still 0.

Note Bryce was very successful in the syntactic approach of applying the definition to construct homomorphisms but did not appear to be thinking about the groups structurally. Interestingly, they did not explicitly acknowledge that the "multiple of 3" map they constructed is the trivial homomorphism and continued to check the homomorphism property for this map.

$\mathbb{Z}$ and $2\mathbb{Z}$. Bryce noticed that both of these groups are infinite but did not use that to say they were isomorphic. Instead, they set up a specific map and appeared to mentally verify that particular pairs of elements would satisfy the homomorphism property while creating their map.

> So they both have an infinite order to them, and I want to say that [$2\mathbb{Z}$] has half the order that the other one does, but when you have infinite orders in your groups that doesn't really matter, so I feel like it could be an isomorphism. I'm not sure how I would set it up, but…[rereading] you'd always have 0 mapped to 0, and then 1 would have to map to 2, cuz that's the next one, and 2 would have to go to 4, so if you just had $f(x)$ where x is in $\mathbb{Z}$, x is an integer, if you map that to $2x$, then it's always going to map to something in $2\mathbb{Z}$. And then 3 would go to 6, so I think that would work for an isomorphism.

Going the other direction, Bryce used $f(x) = x/2$, noticing that this was the inverse of the map previously found. They again noted that "they're infinite, they have the 'same order' [air quotes and sarcastic voice] sort of" and used this to say, "that would make it…one-to-one". Here, they grudgingly accepted the standard mathematical claim that $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality and used this to say the map must be one-to-one, while being uncomfortable with this notion of infinity. This is consistent with their tendency to focus on syntactic verification of the existence of these maps, rather than claiming structural sameness between the groups based on theorems.

For the homomorphism task, going from $\mathbb{Z}$ to $2\mathbb{Z}$, Bryce identified maps involving linear functions with even coefficients, noting that these will not be isomorphisms since they are not one-to-one. When asked to consider whether there would be other homomorphisms, Bryce said:

> I don't think so, because… if you did like instead of $2x$ this was $4x$, then that leaves out 2 and -2, -6, anything that's not divisible by 4, but it has to be one-to-one. So I guess there's no other isomorphisms, but there could be a homomorphism. If you had any other even integer here, then it would be a homomorphism, but it would fail to be one-to-one.

Going from $2\mathbb{Z}$ to $\mathbb{Z}$, aside from the previously discovered isomorphism, they did not find any homomorphisms because "you can't do any other even integer down here [in the denominator] because if you had like 4, then you'd run into problems where you're not mapping to an integer." They seemed to be focused on only using maps that were similar to their isomorphism.

Cayley Tables. For the Cayley table task, Bryce made a comment about the tables needing to "match up" and initially said the groups could not be isomorphic because one group has all self-inverses, and the other does not; but they were reluctant to use the term "inverse" and focused on whether the results of particular computations align with the homomorphism definition.

> So to show an isomorphism, usually the Cayley tables at least match up in a way…I mean they look similar. So everything in this one [the first group] is symmetrical around this diagonal. Everything over here is symmetrical around this diagonal. I don't think it would be an isomorphism because over here every element operated on itself gives back the same thing, so that would be your, I don't want to say inverse… And [in the second group] every element operated on itself doesn't always give you the same element, so I don't think they're isomorphic.

Later, they became unsure of this conclusion and proceeded to count the orders of the elements to check. During this process, they noted that one group was cyclic and the other was not.

> You have an order 4 over here and order 2 over here. So they can't match up. Or you don't have any order 4 over here, so that's why they're not an isomorphism. You could also say this one is cyclic, cuz c would generate the whole group, and this one's not cyclic.

For the homomorphism aspect, they knew that "ultimately it has to respect the group operation" and wrote $f(x * y) = f(x) + f(y)$, but otherwise were unsure how to proceed. In this case, Bryce's syntactic approach focused on definitions and maps was not productive, as they were unsure how to visualize setting up homomorphisms using the Cayley tables.

Blake

$\mathbb{Z}_3$ and $\mathbb{Z}_6$. Blake immediately said that these groups were not isomorphic without giving an explanation. We note this was the fourth of the original set of tasks the students were given, and Blake had previously used the justification of different sizes to explain a lack of isomorphism (for $\mathbb{Z}_5$ to $5\mathbb{Z}$, and $\mathbb{Z}_5$ to $\mathbb{Z}_6$), so we assume they were using the same reasoning here.

Unlike Bryce, Blake found homomorphisms here by considering subgroups: "There is a homomorphism because $\mathbb{Z}$ mod 6 has a three-element subgroup because three divides six so we can just set each element of $\mathbb{Z}$ mod 3 to those elements" [referencing the 0 to 0, 1 to 2, and 2 to 4 mapping written on their paper]. When asked why this would be a homomorphism, Blake struggled to explain, saying "I guess I would have to do it out fully in order to fully convince myself… I would just try to check the $f(ab) = f(a) * f(b)$". However, they never checked this, suggesting that they valued the use of theorems asserting existence more than specific mappings. They also noted the trivial homomorphism would be a homomorphism and believed they had found all homomorphisms "because of the divisors of six and three and matching up the subgroups and stuff." Later in the interview, they were prompted to consider the 0 to 0, 1 to 4, 2 to 2 mapping and reexamine their original map. When asked whether this new map was a homomorphism, Blake again responded with an answer highlighting the structure of the groups.

B: Because 0, 2, and 4 form a subgroup within ℤ mod 6. [Long pause] Because this is isomorphic to ℤ mod 3, so yeah. This looks like it would work. If you add these two, you get the identity back… It looks like I had something along that train of thought before…

I: So is it surprising or not surprising that both of those seem to form homomorphisms?

B: [Blake thinking] I guess it's not surprising because 0, 2, and 4 both form a three-element subgroup of ℤ mod 6 and because three is prime, each of those elements is going to be generators of the whole group, so they're kinda gonna behave similarly. So it's not surprising that you might be able to just switch them around like in this case.

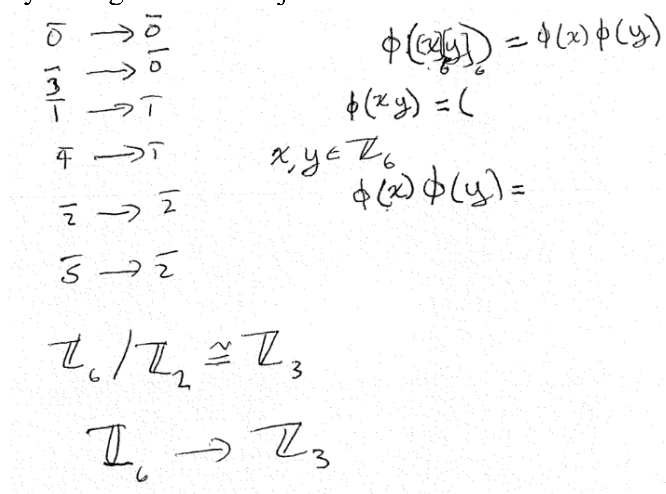

Figure 2. Blake's Homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_3$.

Going from $\mathbb{Z}_6$ to $\mathbb{Z}_3$, Blake again used a semantic approach. They found a map, but when pressed to show it was a homomorphism, they were hesitant and struggled to do this syntactically (see Figure 2). Instead, they invoked the Fundamental Homomorphism Theorem (FHT).

B: There's the trivial homomorphism, there should be another homomorphism as well. Which is…taking the two elements which are order three and mapping it to 1 bar…[writing and thinking] Yeah. I think there is one where you can find two elements in ℤ mod 6 and send them each to one element in ℤ mod 3, but I haven't fully figured out which ones those are, but I think the multiple of 3 will go to 0 bar, and it can work out.

I: Ok. If you can complete that mapping, I would appreciate it.

B: Ok. [writing and thinking] Alright. [turns paper] I just sent all of the elements that are congruent mod 3 to what they're congruent to.

I: Ok. And how do you know that's a homomorphism?

B: Because I know, because of the way modular addition is defined. It should work out pretty nicely in the proof.

I: Just for the sake of doing that once, what would that proof look like?

B: … For some reason, I'm having trouble with this way. But if you go back to the Fundamental Homomorphism Theorem… the kernel of this transformation is isomorphic to $\mathbb{Z}$ mod 2, and $\mathbb{Z}$ mod 6 mod $\mathbb{Z}$ mod 2 is isomorphic to $\mathbb{Z}$ mod 3 by that theorem. Which means that the function $\mathbb{Z}$ mod 6 to $\mathbb{Z}$ mod 3, which has this [air-circling 0 and 3] as its kernel, gives a homomorphism.

The FHT as stated in Blake's textbook is: "Let $f: G \to H$ be a homomorphism of G onto H. If K is the kernel of f, then $H \cong G/K$" (Pinter, 2010, p.151). This theorem is generally used to show the existence of isomorphisms and cannot be used to show a particular map is a homomorphism. Nevertheless, Blake seems to have used the requirements of the theorem to convince themselves that the map they had created would be a homomorphism and provide a kernel as described.

$\mathbb{Z}$ and $2\mathbb{Z}$. Blake said that these groups are isomorphic because they are both infinite and cyclic, remembering that these properties are sufficient to guarantee isomorphic groups: "I've seen the proof that any two cyclic groups of an infinite size are isomorphic, and then I can just find a generator in each one, and then feel reasonably confident in saying they're isomorphic." This is likely the reasoning they used to set up the map where "every integer k will get sent to $2k$" or "you could do negative $2k$". When pressed, they successfully used the formal definition to check that their map satisfied the homomorphism property (see Figure 3).

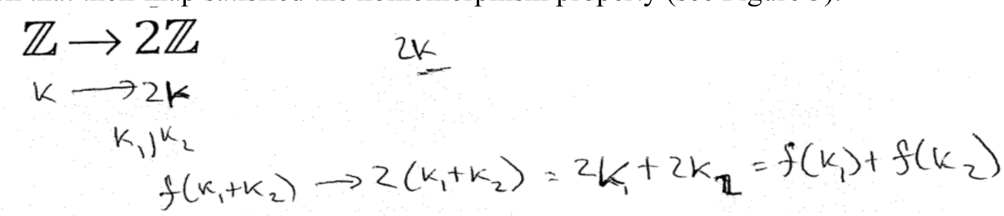

Figure 3. Blake's Isomorphism from $\mathbb{Z}$ to $2\mathbb{Z}$.

Regarding homomorphisms from $\mathbb{Z}$ to $2\mathbb{Z}$, they mentioned "you could probably send them to multiples of 4, or multiples of any other even [number]." Going the other way, they suggested the relation is invertible: "Isomorphism is symmetric so it is indeed isomorphic, I think. You can just take the inverse of that function." They also realized the homomorphisms in this direction will be similar: "So you can construct a function like $2x$ again. You just get multiples of 4 which is again infinite and cyclic, but that should be a homomorphism." Again, they used semantic approaches in setting up homomorphisms, as they mention the image will be infinite and cyclic.

Cayley Tables. Blake, like Bryce, claimed that these groups were not isomorphic because the orders of elements did not align. However, they were different in pointing out the isomorphism class of these groups, stating that one is $\mathbb{Z}_4$ and the other is likely the Klein four-group. From here, they began to look at subgroups in order to find homomorphisms and were generally successful going from $\mathbb{Z}_4$ to the Klein four-group. Blake again attempted to use the FHT for justification, but they were not confident in their answer.

B: So, one of these has to be $\mathbb{Z}$ mod 4 I think.

I: Why is that?

B: …I think that there are only two groups of order 4. I'm not completely certain about that but I think I'm gonna go with that… [writing and thinking] You might be able to map $\mathbb{Z}$ mod 4 onto a two element subgroup of this other one which might be the Klein, or something. I don't really remember. But you could probably choose just a two element subgroup. a seems to be the identity here and each element squared returns a, so yeah. You could just choose like a, b or a, c or a, d and map them [gesture] onto one another.

I: Ok. So how do you know that would work?

B: [Blake writing and thinking]…I'm not sure it's going to work. Because I was going to say something about the Fundamental Homomorphism Theorem but then I remembered that this function then has to be onto. And if we're doing anything like that, we're gonna not hit every element in this group, so I'm not completely confident that'll work.

As we can see, Blake used the structure of the groups, focusing on order of elements and subgroups, to determine a homomorphism, despite their lack of confidence in the mapping.

Discussion

We focused on two students' approaches to three tasks. For the isomorphism tasks, both students correctly applied their knowledge of the properties of the groups to identify whether or not the groups were isomorphic. For $\mathbb{Z}_3$ and $\mathbb{Z}_6$, this amounted to noticing the groups had different sizes, though Bryce expanded to say that this means you cannot create a one-to-one map. For $\mathbb{Z}$ and $2\mathbb{Z}$, both students knew these groups are infinite and were able to set up at least one specific isomorphism in each direction. However, Blake used the fact that both groups are also cyclic to confirm that they should be isomorphic. For the Cayley table task, both students referenced the orders of the elements to justify a lack of isomorphism. However, the proof approaches on the homomorphism tasks were more different. In particular, both semantic and syntactic proof approaches were successfully employed to draw conclusions, and both were linked to struggles at times. Bryce used syntactic approaches to the homomorphism tasks and was both successful and comfortable working with generalized algebraic notation in the definition. In contrast, Blake mostly used semantic approaches to the homomorphism tasks, wherein they focused on the properties and structures of the relevant groups.

While syntactic reasoning generally aligned with instrumental understanding, like Melhuish (2018), we note that semantic approaches did not always seem to suggest a relational understanding. In particular, both students used semantic approaches to finding isomorphisms, but Bryce appeared to be checking memorized properties, more in alignment with instrumental understanding, while Blake seemed perpetually attuned to structure, suggesting a relational understanding. For the homomorphism tasks, Bryce mostly used syntactic approaches in a manner demonstrating instrumental understanding. In contrast, Blake behaved more like the graduate students of Weber (2001) in their semantic approaches, displaying a relational understanding, wherein they intently considered the structures of groups.

We note that the task of finding homomorphisms between groups presented in a Cayley table was particularly difficult. This was also the only task featuring a non-cyclic group. Moreover, while their course had engaged with isomorphism tasks where the groups were presented in Cayley tables, they had not done such a homomorphism task. Rather, this required the students to really understand what homomorphisms do structurally and use properties of the groups since they could not write an algebraic formula for the maps. Thus, instrumental understanding was less likely to be sufficient for this task, and this manifested in only Blake making headway on the homomorphism part of the task. Future research could examine how students' reasoning as portrayed through language relates to their proof approaches and success on tasks, thus binding together Weber and Alcock (2004), Melhuish (2018), and this work with that of Melhuish, Lew, et al., (2020), Rupnow (2021), and Rupnow and Randazzo (2022).

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