Metaphors in Discrete Math: Not All Sameness is the Same

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While sameness is a theme that appears throughout mathematics courses, limited work has examined how multiple types of sameness are understood within the same course. In this paper, we examine survey responses from 49 discrete mathematics students who characterized how they would explain equivalence relations, numerical congruence, and graph isomorphism to a child. Results include language conveying notions of sameness in response to all three prompts but variations in how the sameness was framed for the different concepts. Implications include the need for more work characterizing nuanced differences in students' understandings of similar concepts within specific courses.

Keywords: discrete mathematics, isomorphism, congruence, equivalence relation, sameness

A variety of concepts conveying a notion of sameness have stipulated definitions throughout mathematics and serve purposes that suit their contexts (Rupnow et al., 2022). For instance, isomorphism in abstract algebra and homeomorphism in topology serve similar purposes in conveying structural sameness (in the form of maintaining how elements interact under the defined operations in isomorphism and maintaining continuity in topology). Nevertheless, their definitions differ in order to provide rigor with respect to the relevant objects and to attend to properties that are important to their contexts. However, while some research has examined students' understandings of types of sameness across different courses (e.g., Rupnow et al, 2023), limited work has examined how students attend to multiple types of sameness within the same course context. Thus, we seek to address two research questions:

1. How are discrete mathematics students' characterizations of equivalence relations, congruence, and isomorphism similar?
2. How are discrete mathematics students' characterizations of equivalence relations, congruence, and isomorphism different?

Background Literature

Historically, limited work in mathematics education has attended to sameness as a conceptual area of study; however, notions of sameness have received more attention recently. Students' notions of sameness may impact their understanding of mathematical concepts in ways researchers do not expect, such as claiming multiplication and division are the same binary operation because the same numerical value can be obtained from related operations (Melhuish & Czocher, 2020) or not claiming that having the same graphs indicates having the same functions (Mirin, 2018; Mirin & Zazkis, 2020). However, mathematicians also treat sameness as an understandable concept, including characterizing relative levels of sameness conveyed by concepts (Rupnow & Sassman, 2022). Mathematicians have also highlighted multiple concepts conveying sameness across mathematics, including graph isomorphism, numerical equivalence, and equivalence relations (Rupnow et al., 2022). Much like sameness, equivalence has received more attention of late, including mathematicians' evolving notions of equivalence (Asghari, 2019), characterizations of students' productive interpretations of equivalence (Cook et al., 2021; Cook et al., 2022), and conceptualizations of substitution equivalence (e.g., Wladis et al., 2022).

Moreover, many types of sameness, including equality and congruence, play central roles in the mathematics curriculum. Extensive research on students' understanding of equality has highlighted the value of a relational understanding of equality, in which students understand both sides of the equal sign to have the same value, as this often assists students in understanding algebraic manipulations (e.g., Alibali et al., 2007; Jones et al., 2012; Knuth et al., 2006). Some research has connected students' understanding of sameness to geometric congruence (e.g., Rahim & Olson, 1998; Zazkis & Leron, 1991) or group isomorphism (e.g., Rupnow et al., 2022). Nevertheless, no extant research appears to exist on students' conceptions of equivalence relations, numerical congruence, or graph isomorphism in the context of discrete mathematics.

Theoretical Perspective

Conceptual metaphors is a theoretical perspective intended to provide insight into how individuals' thinking is structured, using their language choices to guide that interpretation (e.g., Lakoff & Johnson, 1980; Lakoff & Núñez, 1997). Cross-domain conceptual mappings connect the cognitive structure of a target concept (e.g., congruence, isomorphism) to the more developed thoughts in source domains (e.g., same properties, classification mechanism). For instance, "An equivalence relation is a classification mechanism" is a conceptual metaphor that gives information about a target domain (equivalence relation) by relating it to a source domain that is already understood in some way (a classification mechanism). We acknowledge that this theoretical perspective imposes the researchers' views on our participants' statements, but believe this lens permits insight into ways of reasoning about our target concepts.

Prior work using conceptual metaphors include examinations of mathematicians' views of isomorphism and homomorphism in abstract algebra (Rupnow, 2021; Rupnow & Randazzo, 2022; Rupnow & Sassman, 2022). Other work has examined undergraduate students' understandings of bases and linear transformations in linear algebra (Zandieh et al., 2017; Adiredja & Zandieh, 2020) and of isomorphism and homomorphism in abstract algebra (Melhuish et al., 2020; Rupnow, 2017). Here we aim to extend the use of conceptual metaphors to other concepts conveying a type of sameness (Rupnow et al., 2022) in discrete mathematics. In particular, we build upon the metaphors identified in Rupnow (2021) and the metaphor clusters identified in Rupnow and Randazzo (2022).

Methods

Data was collected from surveys sent to the Fall 2022 and Spring 2023 sections of the first author's discrete mathematics course. 21 students completed the survey in Fall 2022, and 28 students completed the survey in Spring 2023. We did not observe major differences between the sections and thus combine our reporting into one dataset of 49 students' responses. Students in this course were mostly computer science majors and included freshmen through seniors. Course topics included sets, equivalence relations, functions, modular arithmetic, combinatorics, graph theory, and coding theory. The course textbook was Dossey et al. (2005). Because the instructor's research interests include understandings of sameness in mathematics, equivalence relations, congruence, and isomorphism were all highlighted as types of sameness when those concepts were discussed in class. The surveys were given the last week of classes in each semester, after students had taken in-term exams assessing equivalence relations, numerical congruence, and graph isomorphism. Initial questions on the survey asked about what it means to be the same in math and in discrete math specifically. However, the analysis here focuses on responses to three questions from the middle of the survey:

1. How would you describe the concept of an equivalence relation to a ten-year-old?

2. How would you describe the concept of numerical congruence to a ten-year-old?
3. How would you describe the concept of isomorphic/isomorphism to a ten-year-old?

Data was analyzed by two researchers who coded independently, then discussed and came to consensus for each response. While students were prompted to provide explanations about concepts that could be understood by a child, which itself encourages providing comparative explanations, we (the researchers) sought to classify the underlying meanings conveyed by these explanations according to our interpretations of the participants' responses. Thus, we used the metaphors for isomorphism (in abstract algebra) highlighted in Rupnow (2021) as an initial codebook but added new metaphors when necessary for our data set. This process aligns with codebook thematic analysis (Braun et al., 2019), in which existing codes are used as a basis for coding but new codes are permitted to be added to adequately capture the nuances in data.

Results

We organize the results of our coding into three distinct subsections separated by topic. The first subsection characterizes metaphors utilized to describe the concept of equivalence relation, the second examines metaphors used to describe congruence, and the third depicts metaphors describing isomorphism. In Table 1, we provide an initial overview of the frequencies of each metaphor divided by concept. Metaphors are organized by cluster, with some belonging to sameness-based clusters such as sameness (*generic sameness*, *same properties*, *same but looks different*, *classification mechanism*, *same remainder/leftovers*), sameness/mapping (*renaming/relabeling*, *matching*), and sameness/formal definition (*structure-preservation*) clusters. Other metaphors belong to the mapping cluster (*generic mapping/relation*, *morphing/transformation*, *invertible*), the formal definition cluster (*literal formal definition*, *computation*), or made *no attempt*/*unclear*. Within each cluster, metaphors are arranged by frequency. It should be noted that while 49 responses were examined, the sum of any single column may be greater than 49 due to some students having more than one code per response.

Table 1. Frequencies of metaphors by concept out of 49 total responses.

Metaphor Cluster	Metaphor Code	Equivalence Relation	Congruence	Isomorphism
Sameness	Generic sameness	15	6	9
	Same properties	2	2	19
	Same but looks different	3	3	19
	Same remainder/leftovers	0	20	0
	Classification mechanism	6	0	0
Sameness/ mapping	Matching	0	0	4
	Renaming/relabeling	0	0	1
Sameness/formal definition	Structure-preservation	0	0	2
Mapping	Generic mapping/relation	4	0	1
	Invertible	1	0	4
	Morphing/transformation	0	0	2
Formal definition	Literal formal definition	20	0	0
	Computation	0	8	0
No attempt/ unclear	Unclear/no attempt	6	13	5

Equivalence Relation

The first metaphors we will examine relate to describing the concept of an equivalence relation. For reference, the definition of equivalence relation given in the course textbook was: "A relation on S that is reflexive, symmetric, and transitive is called an equivalence relation" (Dossey et al., 2005, p. 50) and that "A relation R on a set S may have any of the following special properties:

(a) If for each $x \in S$, $x\ R\ x$ is true, then R is called reflexive.
(b) If $y\ R\ x$ is true whenever $x\ R\ y$ is true, then R is called symmetric.
(c) If $x\ R\ z$ is true whenever $x\ R\ y$ and $y\ R\ z$ are both true, then R is called transitive." (Dossey et al., 2005, p. 49)

Overall, the most common metaphor was *literal formal definition*, followed by *generic sameness* and *classification mechanism*. Many responses invoked a mathematically precise definition when asked to describe equivalence relation to a child. The majority of these responses specifically discussed the necessity of being reflexive, symmetric, and transitive. Some achieved this through mathematically rigorous descriptions, such as the following student who used ordered pair notation: "An equivalence relation is a set of ordered pairs that is reflexive (for each x in the ordered pair, there should be (x,x)), symmetric(if (x,y) is in the set, then (y,x) has to be), and transitive(if (x,y) and (y,z) are in the set, then (x,z) must be as well)." Other students described these concepts in a less formal way, utilizing everyday objects and concepts:

> Equivalence relation is a concept that finds the relation between two sets and is reflexive, transitive, symmetric… Imagine reflexive like standing Infront of a mirror, the picture on mirror is the same as a reflection. Symmetric is like taking a paper and drawing a circle and folding it in the middle, now it is a half circle, but when you open it, the symmetric of the half circle is shown making a full circle. Transitive is like saying, Peter and Tom are friends … Tom and Jerry are friends …which will mean Peter and Jerry can be friends.

This response emphasizes the underlying formal structure of an equivalence relation but utilizes visual descriptions, physical action, and common scenarios to convey the key properties involved instead of relying on formal vocabulary and notation.

Other responses characterized equivalence relations using *generic sameness* language. Most of these responses made statements about finding similarity across objects or groups of items: "An equivalence relation is a way of comparing things to see if they are the same in a certain way" or "an equivalence relation is where you can measure the 'sameness' between 2 sets." These metaphors emphasize commonality across elements or sets, while simultaneously conveying the possible variety in what it can mean to be the same.

Another metaphor that many respondents utilized to describe equivalence relations is that of *classification mechanism*. This particular metaphor is unique to equivalence relation, as it describes how elements can be sorted or binned to form distinct groups within an equivalence relation. Many of the responses described grouping everyday objects by some defining characteristic such as color or shape:

> An equivalence relation is a way to group things together based on certain rules. For example, imagine you have a bunch of toys that you want to group based on their color. If you have a blue toy, a red toy, and a green toy, you can group the blue toy with other blue toys, the red toy with other red toys, and the green toy with other green toys. This is an example of an equivalence relation.

Several responses also depicted equivalence relation using *generic relation*. This metaphor was unique to equivalence relation and conveys a relation between elements or groups, such as "A relationship between two or more things that follows some rules." Others provided an *unclear/no attempt* response, of which many simply stated that they were unsure of how to the explain the concept: "It took me a two whole lectures and a Youtube video to piece all of this together myself; tha[t] ten-year-old will leave that conversation more confused than they started."

Congruence

The second group of metaphors relate to congruence. The definition for congruence given in the textbook was: "Let m be an integer greater than 1. If x and y are integers, we say that x is congruent to y modulo m if $x - y$ is divisible by m. If x is congruent to y modulo m, we write $x \equiv y \ (mod\ m)$; otherwise, we write $x \not\equiv y\ (mod\ m)$. We call this relation on the set of integers congruence modulo m" (Dossey et al., 2005, p. 100). Congruence was most frequently described using *same remainder/leftovers*, followed by *computation* and *generic sameness.* Over 25% of respondents provided an unclear response or made no attempt to depict congruence; this was by far the concept with the most responses not attempting to answer the question.

Same remainder/leftovers was the most common metaphor. Many instances of this metaphor were contextless and directly defined congruence, such as "If one number and another number can both be divided by another third number, and they have the same left overs, they are congruent." Other responses introduced context, such as using candy, specific numbers, or cake as in the following: "let's say you have 2 cakes and divide each cake by the same number each cake, if the number of slices left in each cake is the same, then they are congruent." One respondent focused on the sameness of quotients rather than remainders, stating "If you divide both numbers by a single number, and get the same answer." This response was included in the *same remainder/leftovers* category because we felt they intended to describe what was the same numerically, albeit inaccurately.

Just as *same remainder/leftovers* was a metaphor unique to congruence, so too was *computation*. Respondents who conveyed congruence via *computation* often described the action of adding or subtracting a fixed number so as to maintain congruence. In some instances, this was done explicitly: "you take a number and then are able to either add or subtract by another number as many times as you want." Other cases were more implicit, as below:

> Let's say I have a bucket of apples which can only hold 12 apples; when I fill that bucket up, I have to go empty it somewhere (modulo 12). No matter how many apples I put in that "somewhere," I can't put more than 12 apples in my bucket, and I will never keep 12 apples in the bucket (the number is between 0 and 11). So, if I grab 38 apples, there will be 2 apples left in the bucket at the end.

In addition to metaphors unique to congruence, some responses also used *generic sameness* as a metaphor for congruence. Most of these responses discussed congruence in terms of some connection or flavor of sameness across numerical values: "Numerical congruence are the pairs of numbers that are the same as each other in the context of some dimension of numbers."

Isomorphism

The final concept we discuss is isomorphism. The textbook's definition for isomorphism was: "A graph G_1 is isomorphic to a graph G_2 when there is a one-to-one correspondence f between the vertices of G_1 and G_2 such that the vertices U and W are adjacent in G_1 if and only if the vertices $f(U)$ and $f(W)$ are adjacent in G_2. The function f is called an isomorphism of G_1 with G_2 (Dossey et al., 2005, p. 159). Both *same properties* and *same but looks different* were commonly used as a metaphor for isomorphism, as was *generic sameness*. It is notable that metaphors for isomorphism were more focused on appearance than our other two concepts. Isomorphism also had the greatest variety in metaphor codes.

Many of the metaphors discussed shared properties across groups of objects. As isomorphism in Discrete Math is centered on graphs, a large number of these metaphors described graphs with the same properties: "two graphs will need the same number of points to be there, it will also need the same points to be put connected, finally it will need the same number of lines." Other responses emphasized common properties across other 2D or 3D shapes: "imagine you have a shape made out of blue blocks, and another shape made out of red blocks. They look different, but they have the same number of blocks and the same angles between the blocks." This response seems to emphasize notions of geometrical congruence that need not be present in isomorphic graphs (angle measures) as well as notions that apply to graph isomorphism (same number of vertices).

Another commonly used metaphor for isomorphism was *generic sameness*. Focusing first on contexts regarding graphs, the majority of metaphors in this category emphasized the underlying sameness of two graphs: "isomorphic is being of identical or similar form, shape, or structure." One response pulled from a Chemistry perspective, discussing isomorphism as occurring "When two or more crystals have similar chemical compositions exist in the same crystalline form."

Still another metaphor characterizing isomorphism was same *but looks different*. These types of instances might be viewed as a variation on *generic sameness*, but placed emphasis on the idea that while isomorphic objects have the same underlying structure or properties, they may appear different from a visual standpoint. The context of these metaphors varied; as with *same properties*, many discussed graphs: "for two graphs to be isomorphic they are pretty much the same but with a different shape." Others emphasized the underlying action behind these surface differences through a variety of mediums, including play dough and stick figures:

> Let's say you have a stickman, the vertices are the stickman joints such as the knees, elbows, and shoulders, you can pose the stickman to have his arms up, be sitting down, or running, but he will always be isomorphic to the stickman that is just standing in place, because the arms and legs and head are all still connected to the same points.

While not as frequently utilized, there are several additional metaphors that were unique to isomorphism. The physical action of *morphing* or *transforming* from one shape to another was described as a way to demonstrate isomorphism, such as "if you have some play doh, even if you morph the play doh, you still have the same clump of play doh." Others used *matching* language, emphasizing that elements in one object can be directly matched to elements of another:

> For example, imagine you have two puzzles. One puzzle is a picture of a dog and the other puzzle is a picture of a cat. Even though the pictures are different, the puzzles might have the same number of pieces and the same shape for each piece. If you were to switch the pieces between the two puzzles, you could still put them together in the same way. This means the puzzles are isomorphic[.]

Still others described the *structure-preservation* of isomorphism: "an isomorphism is a structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping." Interestingly, both instances of this particular metaphor also included the metaphor *invertible*. There was also one mention of *renaming/relabeling* elements so as to demonstrate the concept of isomorphism, placing emphasis on the arbitrariness and interchangeability of labels in a graph: "If we have two graphs, and they look kinda similar, and we take the names of the points away, and can put them back to make the graphs looks the same, they are isomorphic."

Discussion

This study examined discrete mathematics students' explanations of equivalence relations, congruence, and isomorphism. By using the conceptual metaphor lens, we observed that many explanations focused on sameness, including metaphors from the sameness, sameness/mapping, and sameness/formal definition clusters. Some explanations also highlighted metaphors from the mapping and formal definition clusters or made *no attempt/unclear*. We recognize that students using sameness-infused language is not surprising, given the context. The students were asked about notions of sameness earlier in the survey, and their instructor highlighted each concept as a type of sameness both in class and in the class notes. What we do find interesting is that students highlighted different metaphors for the three concepts. Sameness was salient but conveyed through different common metaphors (or non-attempts) for each concept: *literal formal definition* and *generic sameness* for equivalence relations, *same remainder/leftovers* and *unclear/ no attempt* for congruence, and *same properties* and *same but looks different* for isomorphism.

Then again, perhaps this variation in metaphors should not be surprising, since the instructional emphasis varied in each unit. In the first unit, where equivalence relations were introduced, a key purpose was to familiarize students with the use of mathematical definitions, so extensive time was spent on understanding and verifying whether the reflexive, symmetric, and transitive properties held for each relation. It is possible that this instructional choice obscured the notion of an equivalence relation conveying sameness or serving a classification purpose for some students. For congruence, many students did note the sameness of the remainders, but this also was the concept for which the most students struggled to give a clear description. Perhaps the computation-heavy applications in subsequent lessons (i.e., in the RSA Method) obscured the meaning of congruence itself. Finally, isomorphism had the most variety in metaphors. Notions of sameness came through clearly in students' language for isomorphism, but what was the same varied across lists of properties, a core sameness despite the difference in appearance, and being able to match aspects of different graphs. It is possible that the instructor's more extensive experience in graph theory led to clearer instructional experiences. However, more research is needed on the differential impact of the instructional context versus the concepts themselves.

Additionally, the underlying objects relevant for each concept vary widely in abstractness from almost anything (equivalence relations) to numbers (congruence) to graphs (isomorphism). While previous research has attended to the objects to which sameness-based concepts are applied (Rupnow et al., 2022; Rupnow & Sassman, 2022), this work raises new questions about the direction of the impact of objects and concepts. In particular, do the underlying objects (e.g., graphs) influence our conceptions of the concepts (e.g., isomorphism), the concepts influence our conceptions of the objects, or do the concepts and objects mutually impact each other? For students who later see isomorphism in other contexts (e.g., linear algebra), how might first seeing graph isomorphism impact how they interpret isomorphism of vector spaces?

Finally, the students were prompted to explain in a manner understandable to a 10-year-old. Admittedly, this is an unusual prompt, especially to students who do not intend to teach, but our goal was not to assess the age appropriateness of the explanations. Rather, we believe this prompt permitted a different window into students' understandings than just asking them to describe each concept, as it encouraged a distillation of key idea(s) each student took away. Moreover, it encouraged students to be creative, as displayed in the wide variety of contexts in which students chose to position their explanations (e.g., blocks, candies, Play Doh). Future work can further examine ways to build on the creative examples students highlighted while supporting students to integrate their intuitive understandings with those of the formal definition.

References

Adiredja, A. P., & Zandieh, M. (2020). The lived experience of linear algebra: A counter-story about women of color in mathematics. *Educational Studies in Mathematics, 104*, 239–260. https://doi.org/10.1007/s10649-020-09954-3

Alibali, M. W., Knuth, E. J. Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*(3), 221–247. https://doi.org/10.1080/10986060701360902

Asghari, A. H. (2009). Experiencing equivalence but organizing order. *Educational Studies in Mathematics, 71*(3), 219–234.

Braun, V., Clarke, V., Hayfield, N., & Terry, G. (2019). Thematic analysis. In Liamputtong P. (Ed.) *Handbook of Research Methods in Health Social Sciences*. Springer: Singapore. https://doi.org/10.1007/978-981-10-5251-4_103

Cook, J. P., Dawkins, P., & Reed, Z. (2021). Explicating interpretations of equivalence in measurement contexts. *For the Learning of Mathematics, 41*(3), 36–41.

Cook, J. P., Reed, Z., & Lockwood, E. (2022). An initial framework for analyzing students' reasoning with equivalence across mathematical domains. *The Journal of Mathematical Behavior, 66,* 100935

Dossey, J. A., Otto, A. D., Spence, L. E., & Vanden Eynden, C. (2005). *Discrete Mathematics* (5th edition). Pearson.

Jones, I., Inglis, M., Gilmore, C., & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign. *Journal of Experimental Child Psychology, 113*, 166-176. http://dx.doi.org/10.1016/j.jecp.2012.05.003

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297–312.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by.* Chicago: The University of Chicago Press.

Lakoff, G., & Núñez, R. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In Lyn D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–89). Mahwah, NJ: Erlbaum.

Melhuish, K., & Czocher, J. A. (2020). Division is pretty much just multiplication. *For the Learning of Mathematics, 40*(2), 38–43.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior, 60,* 100806. https://doi.org/10.1016/j.jmathb.2020.100806

Mirin, A. (2018). Representational sameness and derivative. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 569–576), Greenville, SC.

Mirin, A., & Zazkis, D. (2020). Function sameness: Bringing coherence to implicit differentiation. *For the Learning of Mathematics, 40*(3), 14–19.

Rahim, M. H., & Olson, A. (1998). Qualitative patterns in plane geometry. *Journal of Mathematical Behavior, 17*(3), 373–389. https://doi.org/10.1016/S0364-0213(99)80069-X

Rupnow, R. (2017). Students' conceptions of mappings in abstract algebra. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual*

Conference on Research in Undergraduate Mathematics Education Conference on Research in Undergraduate Mathematics Education, (pp. 259–273), San Diego, CA.
Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *Journal of Mathematical Behavior, 62*(2), 100867. https://doi.org/10.1016/j.jmathb.2021.100867
Rupnow, R., & Randazzo, B. (2022). Mathematicians' language for isomorphism and homomorphism. In *Proceedings of the 44th Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 1099–1107). Nashville, TN. https://doi.org/10.51272/pmena.44.2022
Rupnow, R., Randazzo, B., Johnson, E., & Sassman, P. (2022). Sameness in mathematics: A unifying and dividing concept. *International Journal of Research in Undergraduate Mathematics Education.* https://doi.org/10.1007/s40753-022-00178-9
Rupnow, R., & Sassman, P. (2022). Sameness in algebra: Views of isomorphism and homomorphism. *Educational Studies in Mathematics, 111*(1), 109–126. https://doi.org/10.1007/s10649-022-10162-4
Rupnow, R., Uscanga, R., & Bergman, A. (2022). Connecting sameness in abstract algebra: The case of isomorphism and homomorphism. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1131–1136). Boston, MA.
Rupnow, R., Uscanga, R., & Bergman, A. (2023). Variation in students' thinking about sameness and comparisons to mathematicians. In S. Cook, B. Katz, & D. Moore-Russo (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 27–34). Omaha, NE.
Wladis, C., Sencindiver, B., Offenholley, K., Jaffe, E., & Taton, J. (2022). Conceptualizing mathematical transformation as substitution equivalence: The critical role of student definitions. In S. S. Karunakaran & A. Higgins (Eds.), *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 893–900). Boston, MA.
Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics, 95*(1), 21–38. https://doi.org/10.1007/s10649-016-9737-0
Zazkis, R., & Leron, U. (1991). Capturing congruence with a turtle. *Educational Studies in Mathematics, 22*(3), 285–295.