From Conflict to Unity: Leveraging Incoherence in Student Thinking to Improve Understanding

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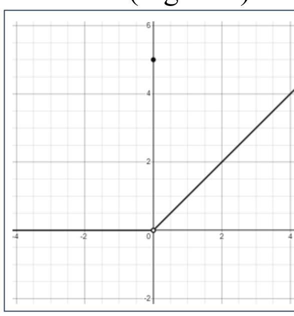
Keywords: schemes, coherence, teaching experiment, proof-texts, continuity, differentiability

Many methodological approaches, such as clinical interviews (Clement, 2000) and teaching experiments (Steffe & Thompson, 2000) are well-equipped for investigating the schemes and constructed knowledge of individuals (Glasersfeld, 1988). Sellers (2020) extended the teaching experiment to the exploratory teaching interview for when the researcher aims to influence student thinking but is not testing a pre-determined hypothesis. Numerous researchers have tasked subjects with reading proofs to probe their thinking (e.g., Dawkins & Zazkis, 2021). I used proof-texts during a teaching experiment which modeled a subject's thinking. This approach was profoundly effective in revealing inconsistencies to the subject in her thinking.

Incoherence as a Basis for Task Design

The lone participant in the study, Rachel, sat for a series of six interviews designed to probe her understanding of the chain rule of differentiation and its underlying ideas, such as continuity, differentiability, and function composition. On several occasions, she evidenced conceptions relevant to the tasks which I conjectured would result in her acknowledging a conflict if presented properly. For example, during the first interview, Rachel stated that a function f was continuous at a if a was in the domain of f. Later in the interview, she said that f was continuous at a if $\lim_{x\to a} f(x)$ existed, regardless of whether a was in the domain of f. Though she did not initially indicate that she found her thinking inconsistent, presenting her with proof-texts which imitated her thinking was effective in helping her recognize conflicts in her thinking and the need to reconcile them (Figure 1).



Claim: f is continuous at $x = 0$.
Proof:
- $\lim_{x\to 0^+} f(x) = 0$
- $\lim_{x\to 0^-} f(x) = 0$
- Since the one-sided limits agree, $\lim_{x\to 0} f(x) = 0$.
- Therefore, f is continuous at $x = 0$.

Claim: f is continuous at $x = 0$.
Proof:
- From the graph, we see that $f(0) = 5$.
- Since f is defined at $x = 0$, f is continuous at $x = 0$.

Figure 1: Graph of Function and Accompanying Proof-Texts

Discussion and Future Steps

The interventions in the teaching experiment were designed only to focus on sources of incoherence (Thompson, 2008). As such, my only desired learning outcome for the experiment was to perturb Rachel's schemes (Glasersfeld, 1988) and observe how she assimilated the new stimuli. Resolving conflicts allowed Rachel to cultivate a much deeper understanding of the target concepts. As such, the methodological combination of exploratory teaching interviews (Sellers, 2020) and proof-texts which imitate the subject's thinking warrants further implementation and investigation.

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Argument-Mirroring Proofs: A Methodological Approach for Helping Students Recognize Incoherence in Their Thinking

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Theoretical Backing

- Radical Constructivism (Glasersfeld, 1988)
  - *Scheme* – a discrete unit of knowledge possessed by an individual
  - *Perturbation* – when expectation associated with a scheme is different from what occurs
  - *Accommodation* – action taken by an individual to account for a perturbation
    - E.g., adjustment of scope of schemes, rejection of schemes, rejecting perturbation as anomaly
  - *Coherence* – the extent to which a thinker's schemes are consistent with one another (Thompson, 2008).

Glossary of New Constructs

- Extensions to tenets of radical constructivism
  - *Disjoint activated schemes* – schemes evoked by individual in response to specific stimuli
    - "Disjoint" refers to notion that individual's schemes may not be related or that individual may not recognize relationships between them
  - *Schemes targeted for conflict* – disjoint activated schemes which an outside observer conjectures represent lack of coherence in subject's thinking
  - *Perceived conflict* – Explicit recognition by the subject that their thinking was indeed incoherent
- ***Argument-Mirroring Proofs*** – texts which use subject's previously exhibited thinking to form and justify a claim, particularly in effort to perturb their schemes

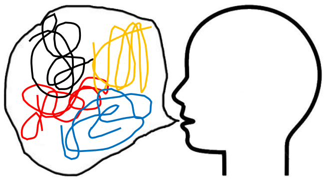
Overview of Case Study

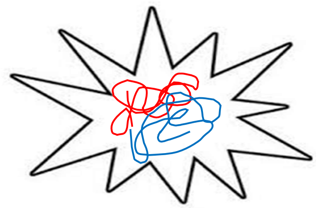
- Participant was an undergraduate student of mathematics education
- Teaching experiment (Steffe & Thompson, 2000)
  - Consisted of clinical interviews (Clement, 2000) and exploratory teaching interviews (Sellers, 2020)
- Task-based interviews focused on continuity, differentiability, and the derivative of composite functions

Argument-Mirroring Proofs, Illustrated

1) Characterize subject's schemes

- Conduct task-based clinical interviews to gather evidence about how student understands mathematical ideas at hand



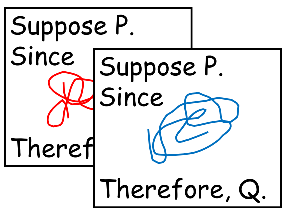


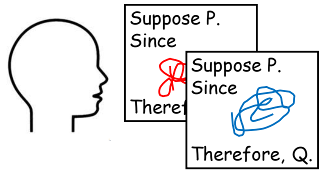
2) Target schemes for conflict

- Code clinical interview data to identify segments of subject's thinking which represent lack of coherence or internal inconsistency

3) Write argument-mirroring proofs

- Construct proof-like texts for use in exploratory teaching interview which model the thinking indicated by the student in the schemes targeted for conflict.



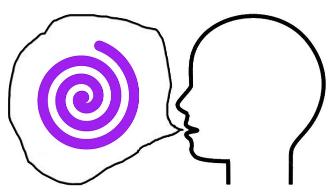


4) Present argument-mirroring proofs

- In an exploratory teaching interview, task the subject with examining the validity of the claims and reasoning in the proof-texts.

5) Characterize subject's schemes

- Code exploratory teaching interview data to gather evidence about how student understanding of mathematical ideas has **changed.**



Interview Tasks

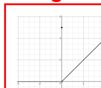
- Clinical interviews
  - Consider where $f(x) = \sqrt{k - 1 - x^2}$ is continuous (resp. differentiable) under various conditions
  - Purpose is to model subject's understanding of the mathematical ideas relevant to the task and target schemes for conflict
- Exploratory teaching interviews
  - Confirm that researcher's model is representative of subject's thinking
  - Present subject with **argument-mirroring proofs** in efforts to perturb their schemes

Illustrative Episode from Study

- Schemes Targeted for Conflict from First Interview

a) If f is defined at a, f is continuous at a.
b) If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$, then f is continuous at a.
c) Although f is not defined where $x = a$, if $\lim_{x\to a} f(x)$ exists, f is continuous at a.

- **Argument-Mirroring Proofs** from Second Interview



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Claim: f is continuous at $x = 0$.
Proof:
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- Schemes Targeted for Conflict after Accommodation

a) $\boxed{\text{If } f \text{ is defined at } a}$, f is continuous at a.
b) $\boxed{\text{If } \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)}$, then f is continuous at a.
c) ~~Although f is not defined where $x = a$, if $\lim_{x\to a} f(x)$ exists, f is continuous at a.~~

$\longrightarrow$ If f is defined at a, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$, and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$ then f is continuous at a.

Discussion and Next Steps

- Key Findings
  - **Argument-mirroring proofs** were unanimously effective in revealing incoherence to subject
  - Subject's thinking became more coherent in response to **argument-mirroring proofs**
- Future Research
  - Conduct similar investigations with students of various mathematical backgrounds
  - Explore other mathematical content areas

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