Factors Influencing Undergraduate Students' Logical (in)Consistency (LinC) in Mathematical Contexts

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This study delves into undergraduate students' ability to self-monitor and organize their claims coherently to ensure logical consistency when evaluating mathematical statements and validating accompanying arguments. To assess the capacity of undergraduate students to maintain logical consistency in mathematical contexts, we designed an online instrument comprising twenty statement-argument pairs, each rooted in mathematical content. We administered this online instrument to 205 undergraduate students, encompassing various levels of proof experience, across three diverse U.S. higher education institutions, including two large public universities and one small liberal arts college. Our analysis reveals a substantial number of undergraduate students displayed logical inconsistencies. It also appears that students may exhibit different levels of logical inconsistencies when the accompanying argument is framed using proof by contradiction, as opposed to direct proof. This prevalence underscores a critical concern in mathematics education, particularly proof-oriented mathematics.

Keywords: logic, logical (in)consistency, statement evaluation, argument validation, proof by contradiction

Our primary aim of this study is to understand logical consistency as it pertains to students' mathematical reasoning and argumentation when evaluating mathematical statements and validating accompanying arguments. By logical consistency in an individual's mathematical thinking, we refer to a state of having no logical contradiction among their assertions. To uphold logical consistency, it is critical that individuals recognize and address any logical contradictions that might remain unnoticed within their assertions. Logical contradictions in mathematics arise when statements conflict, as in false statements such as "x is a multiple of 2 and x is not a multiple of 2." Fundamental to mathematical logic is the law of non-contradiction, which asserts that a statement and its negation cannot both be true simultaneously. For instance, if a student claims both "x is a multiple of 2" and "an argument properly proves x is not a multiple of 2," a logical contradiction arises since the second assertion negates the first, and vice versa. Overlooking such logical contradiction in their reasoning can lead to creating mathematical arguments that the community will not accept. To achieve our research aim, we pose several research questions that guide our investigation:

1. To what degree do students demonstrate logical consistency across a set of statements and accompanying arguments?
2. Is there a correlation between undergraduate students' logical consistency and their exposure to proof-oriented mathematics courses?
3. Do specific contexts or variables influence the likelihood of undergraduate students exhibiting logical inconsistencies (LinC)?

These questions will serve as the foundation for exploring logical consistency in students' mathematical reasoning.

Review of Literature & Theoretical Perspective

To comprehend the significance of logical consistency in mathematical reasoning, we turn to insights from cognitive psychology and educational theory. Cognitive psychologists have long posited the general tendency of people to maintain cognitive consistency and avoid cognitive dissonance in various contexts (e.g., Abelson et al., 1968; Festinger, 1957). The principle of maintaining cognitive consistency has been applied to interpersonal relations, beliefs, feelings, and actions (Cooper, 1988; Cvencek et al., 2014; Gawronski & Strack, 2008). However, the relevance to mathematical reasoning remains intriguing. Our study extends this notion into the realm of mathematical thinking, which we refer to as logical consistency. It is essential to recognize the unique challenges that mathematical reasoning presents, particularly concerning logical contradiction, while evaluating mathematical statements and validating accompanying arguments (Roh & Lee, 2018).

Our study is grounded in a theoretical framework that draws from cognitive psychology and constructivist learning theory. Cognitive psychologists have argued that individuals naturally strive to maintain cognitive consistency and resolve cognitive conflicts (e.g., Piaget, 1967). In learning, cognitive conflicts are considered essential for individuals to construct new knowledge or modify their existing knowledge structures (Glasersfeld, 1995). This perspective suggests that students actively seek logical consistency in their mathematical reasoning and argumentation. Research showcases that students can also enhance their reasoning by addressing logical inconsistencies in their mathematical arguments (Dawkins & Roh, 2016; Ely, 2010; Roh & Lee, 2011, 2017).

However, can students effectively identify logical contradictions within their mathematical assertions when they exist? Roh and Lee's study (2018) reported that more than 20% of undergraduate students ($n = 47$) who completed introductory proof courses displayed logical inconsistencies in their assertions. This suggests that even with substantial mathematical content knowledge and mathematical proofs, logical inconsistencies (LinC) may persist in students' mathematical reasoning. Building upon Roh and Lee's (2018) research, we extend our investigation to assess the frequency with which students exhibit logical inconsistencies and identify the prevalent types of such inconsistencies when students evaluate mathematical statements and validate accompanying arguments.

Methods

The LinC Instrument

We developed the LinC instrument to assess students' logical (in)consistency when evaluating a statement and validating an accompanying argument. The current version of the LinC instrument focuses on content areas in precalculus and calculus, employing two factors for statements (logical complexity and truth-value) and three factors for accompanying arguments (attempts, validity, and frames). Table 1 presents the characteristics of the final 20 statement-argument pairs derived from these factors.

Table 1 Summary of the LinC items (20 items in total)

Quantifiers in the Statement	Truth-Value of the Statement		An Attempt in the Argument		Validity of the Argument		A Frame of the Argument		
	True (10)	False (10)	Proof (10)	Disproof (10)	Valid (12)	Invalid (8)	Direct (4)	by Contradiction (7)	by Example (9)
$\forall$ (4)	2	2	3	1	2	2	1	1	2
$\exists$ (6)	3	3	2	4	4	2	1	2	3
$\forall\exists$ (4)	2	2	2	2	3	1	0	2	2

$\exists\forall$ (6)	3	3	3	3	3	3	2	2	2

Each LinC item starts with a statement-argument pair, followed by three questions: (1) evaluate if a given statement is true or false; (2) determine if a given argument is an attempt to prove or an attempt to disprove the given statement; and (3) evaluate if the given argument proves properly what it attempts to prove. Figure 1 illustrates an example of LinC items.

Item 1. Consider the statement and argument below:

Statement: For all $x > 0$, $x^3 - x^2 + x > 0$.
Argument: Suppose there exists $x > 0$ such that $x^3 - x^2 + x \leq 0$. Then, since $x > 0$ and $x^3 - x^2 + x \leq 0$, $x^2 - x + 1 \leq 0$. However,

$$x^2 - x + 1 = \left(x^2 - x + \frac{1}{4}\right) + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0.$$

1. Check the most appropriate one about the statement.
   (a) The statement is true.
   (b) The statement is false.
   (c) We cannot determine if the statement is true or false.
2. Check the most appropriate one to describe what this argument attempts to prove.
   (a) This argument attempts to prove the statement is true.
   (b) This argument attempts to prove the statement is false.
   (c) We cannot determine if this argument attempts to prove the statement is true or false.
3. Check the most appropriate one to describe the validity of this argument.
   (a) This argument proves properly what it attempts to prove.
   (b) This argument does not prove properly what it attempts to prove.
   (c) We cannot determine whether or not this argument proves properly what it attempts to prove.

Figure 1 An example of the LinC items

In each LinC item, we first asked students about the truth value of the given statement before asking them to evaluate the accompanying argument. However, students were allowed to change their answers to any of these three questions before moving to the next LinC item. As a result, students could determine or revise their determination of the truth value of the statement after engaging with the second and third questions about the accompanying argument. That is, students had opportunities to self-organize their thinking to have no logical contradiction among their responses to the three questions given in each LinC item. Thus, if there is a logical contradiction among a student's response to the three questions, we interpret the student's response as displaying logical inconsistency.

For example, a student might initially believe that the statement is true. Still, after reading the given argument about the statement, the student might construe and accept that the argument proves properly the statement is false. Suppose the student recognized the logical contradiction between their initial evaluation of the statement as true and their subsequent validation of the argument that proves the statement false. In that case, they may reconsider either their evaluation of the statement as false or their validation of the argument to maintain non-contradiction in their responses to the LinC item.

However, some students may not detect a logical contradiction even if it exists in their responses. Then, they would not experience psychological pressure to self-organize their thinking to have no logical contradiction, and consequently, they would not change their responses. In this case, the presence of logical contradictions in students' responses suggests that the student exhibits logical inconsistency. That is, the students cannot self-organize their thinking to have no logical contradiction. The three questions in each LinC item allow logical inconsistencies in students' responses to manifest if they exist. The LinC instrument's reliability is acceptable with Cronbach's α=0.686 (n=205).

Data Collection

In this study, we collected data via an online survey utilizing the LinC instrument. The survey was administered at two large public universities and one small liberal arts college in the United States from fall 2019 to spring 2022. We obtained responses from a total of 205 undergraduate students who dedicated sufficient time to complete the LinC instrument. Among them, 86 participants were recruited from in-person classes during fall 2019, fall 2021, and spring 2022. The rest of the participants (119) were from online classes during spring 2020 and fall 2020, coinciding with the COVID-19 pandemic, during which educational institutions experienced a rapid shift towards remote and online teaching modes.

Data Analysis

Table 2 provides a comprehensive list of all possible logical inconsistencies that could arise from evaluating a statement and validating an argument about the statement. This table serves as a valuable reference tool for analyzing logical inconsistencies in students' responses to the LinC instrument and identifying patterns in students' responses.

Table 2 All instances of responses identified as displaying logical inconsistencies to the LinC items

LinC Type	Response to Question (1)	Response to Question (2)	Response to Question (3)
aba	The statement is **True.**	The argument attempts to prove the statement is **False.**	The argument proves **properly** what it attempts to prove.
aca	The statement is **True.**	We **cannot determine** what it attempts to prove.	The argument proves **properly** what it attempts to prove.
baa	The statement is **False.**	The argument attempts to prove the statement is **True.**	The argument proves **properly** what it attempts to prove.
bca	The statement is **False.**	We **cannot determine** what it attempts to prove.	The argument proves **properly** what it attempts to prove.
caa	We **cannot determine** if the statement is true or false.	The argument attempts to prove the statement is **True.**	The argument proves **properly** what it attempts to prove.
cba	We **cannot determine** if the statement is true or false.	The argument attempts to prove the statement is **False**.	The argument proves **properly** what it attempts to prove.
cca	We **cannot determine** if the statement is true or false.	We **cannot determine** what it attempts to prove.	The argument proves **properly** what it attempts to prove.

In order to examine the first research question, we report descriptive statistics broken down at the item level. To examine the second research question, we considered a level of proof experience (in terms of the number of courses) as a predictor of the LinC overall score. We created a linear model and a hierarchical linear model (students nested within teachers), finding that the nesting added no explanatory power. Finally, for the third research question, our analysis focused on a subset of the data of 86 participants. This subset was drawn from the data collected during which classes were delivered through an in-person teaching mode. Note that the year 2020 coincided with the COVID-19 pandemic, and we acknowledge that the experience, perceptions, and academic performance of participants in online teaching mode during the pandemic may have been influenced by the unique circumstances and challenges posed by the pandemic. Focusing on the subset of participants who experienced in-person teaching, we aimed to mitigate the potential confounding variables influencing the third research question.

Results

RQ1: To what degree do students demonstrate logical consistency across a set of statements and accompanying arguments?

Of the participants ($n = 205$), 81% (187 students) displayed logical inconsistencies at least once while responding to the 20 items in the LinC instrument. Among those who showed logical inconsistencies, 73% displayed them at least twice, 64% at least three times, 53% at least four times, and 40% at least five times. These findings suggest that many undergraduate students in mathematics courses display logical inconsistencies in their thinking. These findings also highlight the need for further research and instructional interventions to address this issue.

RQ2: Is there a correlation between undergraduate students' logical consistency and their exposure to proof-oriented mathematics courses?

In order to explore this relationship, we developed a series of linear models. The first model included both in-person and online semester data. TeachingMode is 1 when online and 0 when in person. ProofExperience takes on a value of 0 for no-proof courses, 1 for one proof course, and 2 for two or more proof courses. We estimated:

$$LinC\ Score\ =\ 18.22 - 3.92 * TeachingMode + 0.13 * ProofExperience$$

finding that TeachingMode was significant with $p < .001$, but ProofExperience was not significant, $p = 0.58$. Since TeachingMode plays an unclear role (perhaps relating to modality or pandemic effect), we also ran a model on just the in-person data, finding similar results. The estimated coefficient for proof experience was 0.05 with a $p = 0.89$. These results were rather robust, with proof experience not being significant in any model our team ran. This indicates a surprising result: Logical inconsistency does not appear related to the amount of proof and formal representation system exposure.

RQ3: Do specific contexts or variables influence the likelihood of undergraduate students exhibiting logical inconsistencies (LinC)?

Because the prior analysis showed that teaching mode had a substantial impact on students' performance, we focus this early preliminary exploration of RQ3 on the in-person subset. Of the participants drawn from the in-person teaching mode ($n = 86$), 57% (49 students) displayed logical inconsistencies at least once while responding to the 20 items in the LinC instrument. Among those who showed logical inconsistencies, 40% displayed them at least twice, 30% at least three times, 15% at least four times, and 5% at least five times.

We examined if these students are more likely to display logical inconsistencies with certain types of statements or arguments. In particular, we analyzed the student responses considering the statements' complexity and the frames of the arguments as the factors that might affect students' logical inconsistencies. In Tables 3 and 4, the LinC frequency for each item represents the number of students displaying logical inconsistency in their responses to that specific item.

LinC and Statements' Logical Complexity. We may assume that students would have greater difficulty understanding mathematical statements with more complex logical structures, such as multiply quantified statements. Similarly, we might anticipate students encountering logical inconsistencies more like in statements with greater logical complexity. We might expect more students to display logical inconsistencies when engaging with multiply quantified statements in the LinC items 11-20 than statements involving a single quantifier in the LinC items 1-10. However, our analysis yielded unexpected results. Students' responses to LinC items contained logical inconsistencies, regardless of the logical complexity of the statements. Over 10% of the students displayed logical inconsistencies on each of the LinC items, including those with complex statements involving two quantifiers (items 11, 15, 17, and 20) and even those with relatively simple statements involving one quantifier (items 1, 2, 6, and 9) as detailed in Table 3. This finding suggests the frequency of logical inconsistencies of students does not significantly differ with respect to the logical complexity of the given statements.

LinC and Frames of Arguments. On the other hand, the type of argument frames points to potential differences in the frequency of logical inconsistencies. We observed a higher occurrence of logical inconsistencies when the argument was framed using proof by contradiction instead of direct proofs (see Table 4). To be specific, over 10 % of the students displayed logical inconsistencies on all but one LinC item with the seven arguments framed using proof by contradiction (e.g., LinC item 11), whereas none of the LinC items with the four arguments framed using direct proofs (e.g., LinC item 8). We can conjecture that the contradiction frame may relate to how students attend to logical consistency.

Table 3 LinC Frequency by Statement Types (n=86)

Item	LinC Frequency	Statement	
		T/F	Quantifier
1	10	T	∀
2	13	F	∀
3	0	T	∀
4	5	F	∀
5	1	T	∃
6	9	T	∃
7	4	T	∃
8	4	F	∃
9	10	F	∃
10	5	F	∃
11	15	T	∀∃
12	2	T	∀∃
13	5	F	∀∃
14	6	F	∀∃
15	10	T	∃∀
16	3	T	∃∀
17	10	T	∃∀
18	2	F	∃∀
19	8	F	∃∀
20	12	F	∃∀

Table 4 LinC Frequency by Argument Types (n=86)

Item	LinC Frequency	Argument		
		Attempt	Validity	Frame
1	10	Prove	Valid	Contradiction
6	9	Prove	Valid	Contradiction
11	15	Prove	Valid	Contradiction
15	10	Prove	Valid	Contradiction
9	10	Disprove	Valid	Contradiction
14	6	Disprove	Valid	Contradiction
20	12	Disprove	Valid	Contradiction
4	5	Prove	Invalid	Direct
16	3	Prove	Invalid	Direct
8	4	Disprove	Valid	Direct
19	8	Disprove	Valid	Direct
3	0	Prove	Invalid	Example
12	2	Prove	Invalid	Example
18	2	Prove	Invalid	Example
5	1	Prove	Valid	Example
7	4	Disprove	Invalid	Example
10	5	Disprove	Invalid	Example
17	10	Disprove	Invalid	Example
2	13	Disprove	Valid	Example
13	5	Disprove	Valid	Example

Conclusion & Discussion

Our study investigated the factors influencing logical inconsistencies among undergraduate students while evaluating mathematical statements and validating accompanying arguments. Initially, we expected students to struggle more with mathematical statements with greater logical complexity. However, our findings challenge this assumption. Surprisingly, students exhibited logical inconsistencies regardless of the logical complexity of the statements. On the other hand, students may exhibit different levels of logical inconsistencies when the accompanying argument is framed using proof by contradiction, in contrast to direct proof. The complexities involved in indirect proofs (e.g., Antonini & Mariotti, 2008) may be related to logical inconsistencies.

Mathematical reasoning relies fundamentally on avoiding logical inconsistencies, as even a single logical inconsistency can render an entire argument invalid. Our study reveals that a substantial number of undergraduate students displayed logical inconsistencies. This prevalence underscores a critical concern in mathematics education, particularly proof-oriented mathematics. We believe that identifying and addressing logical inconsistencies in students' thinking should be a top priority for promoting their learning of proof-oriented mathematics. Therefore, it is imperative for mathematics educators and researchers to carefully monitor students to see when logical inconsistencies exist in students' thinking and understand the factors that affect their occurrence.

Our study has limitations, including the specific context (precalculus and calculus) and sample size. Future research could explore logical consistency in diverse mathematical contexts and involve larger and more diverse student populations. Additionally, the high frequency of logical inconsistencies among undergraduate students highlights a potential gap in their mathematical reasoning skills. The frequency of these logical inconsistencies suggests a pressing need for further research and instructional interventions to address this issue. This is especially important as traditional proof courses do not seem to serve our students in developing their logical consistency. Investigating the effectiveness of specific instructional interventions in addressing logical inconsistencies is a promising avenue for future research.

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