The Cognitive Obstacles Associated with Structuring for Mathematization Undergraduates Encountered During Dynamic Modeling Tasks

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Studies have described a number of blockages to mathematizing, the process of transforming a real-world situation into a mathematical model. Recently, researchers have documented four cognitive obstacles associated with those blockages in physics-based modeling tasks. We attempted to extend these findings to biological-based modeling tasks. However, we found it difficult to use the theoretical lens as described in the previous literature. This difficulty arose because the previous theoretical lens required the delineation of real-world objects and mathematical objects, a pursuit recently shown to be unattainable in some contexts. By examining students mathematization under a quantitative reasoning and symbolic forms lens, we were able to find two cognitive obstacles analogous to the ones found in previous research, confirming their existence with a different demographic of students and different task scenarios.

Keywords: modeling, quantitative reasoning, differential equations

The idea of using mathematical modeling to enrich STEM education is well established. It is an idea upheld by the Common Core State Standards (CCSSI, 2010), and researchers alike (Blum & Niss, 1991). One line of inquiry into how students learn to model focuses on identifying and improving the competencies students need to make progress in a modeling problem (Schukajlow et al., 2018). The competency which has received most attention is *mathematizing* – constructing a mathematical representation of the mental model of the scenario. It is particularly difficult for students (Brahmia, 2014; Galbraith & Stillman, 2006; Stillman & Brown, 2014). Because each phase of modeling builds on decisions the modeler made earlier, some of the difficulties students encounter carrying out mathematizing come from how they structured the real-world scenario. In order to understand why some students experience blockages when they work on modeling problems, researchers have begun to catalog the cognitive obstacles associated with structuring for mathematizing that arise when students work on modeling problems.

Literature Review

Mathematical models are mathematical systems that represent real-world systems. They are conceptual systems "consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)…[A model] focuses on structural characteristics of the relevant system" (Lesh & Doer 2003 p. 10). Historically, scholars have conceptualized the process of generating the representational system as an idiosyncratic cycle that can be disrupted by inadequately performing key activities to transition from one competency to another. Studies have described a number of blockages to mathematizing such as representing elements mathematically so known formulae can be applied (Galbraith & Stillman, 2006) and failing to define variables or failing to realize interdependences among variables (Klock & Siller, 2020). These blockages should be understood as arising from structural choices about which elements, relations, and operations are relevant. Recent studies affirm that lack of content knowledge was not the root of students' difficulties during modeling, but rather a twopart set of skills: structuring the scenario and then translating the structure into a mathematical representation (Jankvist & Niss, 2020; Niss, 2017). Niss (2017) observed four primary obstacles to mathematizing that could be attributed to cognitive processes. He framed mathematizing as making simplifying assumptions about objects in the real world that are specified by the modeler and generating relationships between those objects that are relevant to solving the real-world problem. From this perspective, student blockages could be categorized as (a) not identifying a mathematical object to impose on the scenario (b) not identifying a key real-world object to impose onto the scenario (c) imposing unhelpful mathematical objects, and (d) making structural choices that led to a mathematical representation that students were unable to work on (e.g., choices that would require too-advanced mathematics). In concluding remarks, Niss called for other researchers to corroborate and expand upon this list of cognitive obstacles in other content areas. A call we set out to respond by extending Niss's framework to undergraduate STEM majors' work on canonical modeling problems from biology, including disease transmission and predator-prey dynamics. We found that Niss's framework depends on being able to reliably identify and distinguish between a mathematical object and a real-world object, which was challenging to do empirically in the biological contexts since it was not obvious what the relevant mathematical or real-world objects would be. We were unable to understand what the cognitive obstacles associated with structuring for mathematizing from this perspective. Recently, Zimmerman et al., (2023) argued that there is no clear distinction between mathematical objects and real-world objects, shedding light onto limitations of Niss's framework for settings outside of physics problems. Thus, a promising approach to understanding the difficulties students face while mathematizing would need to absolve the researcher from assuming a distinction between real-world object and mathematical object.

To address this theoretical and methodological need, we adopt constructs from quantitative reasoning (QR) and symbolic forms. Quantitative reasoning is the conceptualization of a situation into a network of quantities and quantitative relationships (Thompson, 1993). A quantity is a triple of an object, attribute, and quantification (Thompson, 2011). Quantification means to conceptualize an object with a measurable attribute so that the measure is proportional to its unit (Thompson, 2011). A quantity is made from an individual's conceptions of objects meaning a quantity is idiosyncratic to the individual (Ellis, 2007). An example of a quantity in a disease transmission context could be the number of sick people at time t. The object is the sick population, the attribute is amount, and evidence of quantification could be a student explaining they could feasibly measure the number of people on one day by counting them. A quantitative operation is a conceptual operation where an individual creates a new quantity in relation to already created quantities (Ellis, 2007; Thompson, 2011). Symbolic forms are a type of cognitive resource comprising of two components: a symbol template and a conceptual schema (Sherin, 2001). For example, a common conceptual schema for the symbol template $[\,] \times [\,]$ is iterating one quantity by the magnitude of another (i.e., repeated addition). In this way, a mathematical model is a conceptual system encompassing a modeler's ideas they find relevant about a real-world system (e.g., a fish tank, or an island habitat). The elements of the model are quantities the modeler imposes onto the scenario. The relationships between elements, operations, and rules governing interactions are determined by the modeler's QR. A modeler externalizes quantitative operations by selecting the symbol template whose conceptual schema comports with the quantitative operation. For example, a modeler could multiplicatively combine number of sick people at time t and number of healthy people at time t to create the quantity number of interactions between sick and healthy people at time t. The modeler might externalize that quantitative operation via the symbol template [ ] × [ ], because repeated addition comports with how they multiplicatively combined the two quantities. In Niss' (2017) framing, cognitive obstacles arose from incongruencies in identifying real-world objects, and assigning the appropriate mathematical objects to those real-world objects. In our framing, cognitive obstacles arose from imposing quantities onto the scenario, creating new quantities by combining other quantities, and expressing that combination externally. From this perspective, we pose the question: What cognitive obstacles to mathematizing, arising from structuring, do students encounter when working on dynamics modeling problems?

Methods

This qualitative study draws data from a larger design study of facilitator scaffolding moves that foster undergraduates' modelling competencies. In this paper, we report on 12 participants' work on three canonical modelling tasks: a predator-prey dynamics scenario, a contaminated tank scenario, and a disease transmission scenario. Participants were volunteers from STEM courses who stated they had some familiarity with mathematical concepts like instantaneous rate of change and reported getting at least C's in their mathematics course work. The task design and task-based interview protocols attended to participants' QR and intentionally focused on similarities of conceptual schema across task contexts. In all three tasks, participants aimed to write a (system of) differential equation(s) modeling the dynamics of the scenario. In the predator-prey task, students were asked to create two differential equations that modeled the rate of change of cats and birds accounting for their interactions. In the contaminated tank task, a buffering agent is mixed with water to create a buffering solution. Students were asked to model how quickly buffering agent in the tank changed given the buffering solution entered the tank at 5 liters per minute at a concentration of $1 - e^{-\frac{t}{20}}$ grams per liter. In the disease transmission task, participants were asked to model how quickly a disease was spreading through a community of susceptible, infected, and removed populations.

Analysis of recordings of and written work from the interviews proceeded in four stages. First, we generated a list of quantities each participant imposed onto the task scenario by describing the object, attribute, and how the participant exhibited quantification for that attribute according to the quantification criteria developed by Czocher and Hardison (2021). Second, we created a chronological narrative documenting how participants combined quantities and externalized that combination using symbolic forms. Third, we identified instances where a participant's progress in the task halted. Some key indicators were: the participant said they were stuck, the participant wrote down two mathematical expressions for the same quantity (or quantitative relationship) and said one of the expressions should be correct, or the interviewer noticed that what the participant wrote down did not mathematically mean what the participant was explaining. In each instance, we inferred the cognitive obstacle that impeded the participant's progress based on the quantities and quantitative operations previously documented. Finally, we looked across instances for commonalities which we report as types of cognitive obstacles below.

Results

Across the three canonical modeling problems and participants, we observed many blockages to mathematization that could be attributed to cognitive processes and report on two with origins in structuring: an unhelpful quantitative relationship took precedent, and a key quantity was not salient for the participant. A third type of cognitive obstacle (a participant's specific quantification of a quantity did not permit them to sensibly combine it with other already-present quantities using well-known formulas) was previously reported in Roan and Czocher (under review). We illustrate each kind of cognitive obstacle to evidence its existence and its role in connecting structuring to mathematizing. To evidence the two types of cognitive obstacles existence, we selected instances that best exemplified a specific type of cognitive obstacle. To do that will show work from two students, Ivory (Physics) and Niali (Electrical Engineering).

An unhelpful quantitative relationship took precedent.

This cognitive obstacle occurred when a participant focused on externalizing a quantitative relationship that was not helpful (from the interviewer/researcher perspective) in creating a differential equation that modeled the scenario. To evidence this type of cognitive obstacle, we show work from Ivory (Physics) on the disease transmission task. To start, we show the quantities, quantitative relationships and the corresponding symbols and expressions she had created up to the point of interest in Table 1.

Table 1. Ivory's quantities, quantitative relationships, and the corresponding symbols and expressions.

Quantity or Quantitative relationship	Symbol or Expression
Amount of susceptible people at time t	$S(t)$
Amount of infected people at time t	$I(t)$
Amount of removed people at time t	$R(t)$
Mortality rate	d
Accumulated amount of dead people at time t	$m(t)$
Amount of recovered people at time t	$r(t)$
Amount of dead people at time t and amount of recovered people are two disjoint subsets of amount of removed people at time t	$r(t) = 1 - m(t)$

Ivory was working on creating an expression for *amount of dead people at time t*. She noted some pertinent information she needed to consider, namely, that "there's a two-week delay" between symptom onset and death "…because if you haven't died within two weeks, you've recovered, is what we're assuming here." Ivory assumed that (A) the *amount of dead people at time t* depended upon *amount of infected people at time t, 14 days ago*. We identified that Ivory had stopped progressing towards creating a differential equation for the scenario because she wrote down multiple mathematical expressions for the same quantitative relationship, as if searching for the correct mathematical expression. Table 2 shows the different expressions Ivory tried and then rejected to depict quantitative relationship (A).

Table 2. Ivory's quantities, quantitative relationships, and the corresponding symbols and expressions.

Quantity or Quantitative relationship	Symbol or Expression
amount of dead people at time t depended upon amount of infected people at time t, 14 days ago.	$m(t) = I(t + 14) \times d$
	$m(t) = \frac{dS}{d(t + 14)}$
amount of removed people at time t depended upon amount of infected people at time t, 14 days ago.	$\frac{dR}{dt} = \frac{dI}{d(t - 14)}$
	$\frac{dR}{dt} = \frac{dS}{d(t - 14)}$
	$\frac{dR}{dt} = I(t - 14)$

Ivory experienced a cognitive obstacle related to her attempts to depict a quantitative relationship she did not know how to represent adequately. In Ivory's work, the quantitative relationship (A) took precedence over more helpful relationship like (B) positive flux in the removed population is proportional per unit time to the current infected population. She could not adapt her models for relationship (B) because they did not account for relationship (A) in an explicit way. Neither did she think her models accounting for (A) were adequate. She reflected she was "still caught up on the 14-day lag thing." We assert that this was a cognitive obstacle for Ivory because she was not able to meet her goal (to create a system of differential equations that modeled the disease spread) until this quantitative relationship lost precedence.

A key quantity was not quantified by the modeler.

This type of cognitive obstacle occurred when a key quantity was missing from the participant's structure so other key quantities could not be constructed. Across our participants and tasks, this cognitive obstacle seemed to occur when participants thought they could not use a quantity in an expression because they thought they needed to find an equation to determine it (a way to measure its value) first. To evidence this type of cognitive obstacle, we show work from Niali (Electrical Engineer) on the contaminated tank task. To start, we show the quantities, quantitative relationships and the corresponding symbols and expressions he had created up to the point of interest in Table 3. Niali stated that he needed to find a mathematical expression for the quantity concentration of buffering exiting, as evidenced in the exchange below:

Niali: OK, we have a-- let's see. Have a fluid in the tank. We have a tank. We have a pump which adds water with concentration. The water starts full. Uh, buffer enters through here, and out comes the water. The water exits at a rate of 5 liters per minute. It enters 5 liters per minute at a rate of-- that-- oh, what is this, grams of buffer per liter. So what does it exit at?

Interviewer: And by exit at, you're talking-- you're thinking specifically about the concentration, the true concentration of the liquid exiting, right?

Niali: Yes. So how many grams per liter of the chemical is exiting?

Table 3. Nilai's quantities, quantitative relationships, and the corresponding symbols and expressions.

Quantity or Quantitative relationship	Symbol or Expression
Rate of buffering solution entering in liters per minute	$water\ in = 5$
Concentration of buffering solution entering in grams per liter	$1 - e^{-\frac{t}{20}}$
Rate of buffering agent entering in grams per min	$5 \times (1 - e^{-\frac{t}{20}})$
Rate of buffering solution exiting is the same as the rate of buffering solution entering	$water\ out = 5$
Maximum volume of the tank in liters	300

We identified that Niali had stopped making progress towards creating a differential equation for the scenario because he wrote down multiple mathematical expressions for the same quantity, evidencing an ongoing search for an adequate one. In Table 4, we showcase all the different expressions Niali tried and rejected to depict the quantity *concentration of buffering exiting*.

Table 4. Nilai's quantities, quantitative relationships, and the corresponding symbols and expressions.

Quantity or Quantitative relationship	Symbol or Expression
Concentration of buffering solution entering is the same as concentration of buffering solution exiting	$\%\ concentration\ in = 1 - e^{-\frac{t}{20}} = \%\ concentration\ out$
Concentration of buffering solution exiting	$\frac{5 \times (1 - e^{-\frac{t}{20}})}{300}$
	$\left(\frac{5 \times \left(1 - e^{-\frac{t_{Final}}{20}}\right)}{300} - \frac{5 \times \left(1 - e^{-\frac{t_{iniital}}{20}}\right)}{300}\right) \times \frac{1}{2}$

Niali experienced a cognitive obstacle related to his attempts to create a quantity when he did not have the quantities available to him to do so. In Nilai's work, the quantity *concentration of buffering exiting* could not be constructed until the quantity *amount of buffering agent in the tank* was quantified. We assert that this was a cognitive obstacle for Niali because he was not able to meet his goal (to create a differential equation that modeled how quickly the buffering agent in the tank was changing) until the quantity *amount of buffering agent in the tank* was quantified. Further, other students in the study who did explicitly quantify *the amount of buffering agent in the tank* did not encounter this obstacle.

Discussion

In this paper, we found two cognitive obstacles associated with structuring for mathematizing students encountered in three canonical modelling tasks: a predator-prey scenario, and a contaminated tank scenario. These were (1) an unhelpful quantitative relationship took precedent, and (2) a key quantity was not quantified by the modeler. While not all participants experienced each type of cognitive obstacle, no one obstacle was specific to one task. This implies that the cognitive obstacles we showcase here are not a consequence of the task scenario but are rooted in how students' reason quantitatively while mathematizing. We set out in this work to confirm the cognitive obstacles found by Niss (2017) existed in these different task scenarios with a different demographic of student. However, due to the nature of the tasks we selected, we found it difficult to delineate real-world objects and mathematical objects. This difficulty has been empirically shown to occur in physics as well (Zimmerman et al., 2023). To continue our investigation, we elected to investigate these cognitive obstacles through a quantitative reasoning and symbolic form lens. We were successful in finding analogous cognitive obstacles found by Niss (2017). The cognitive obstacle an unhelpful quantitative relationship took precedent is analogous to Niss (2017) cognitive obstacle "making structural choices that lead to a mathematical representation that students were unable to work on". However, here Ivory made structural choices that she could not translate into a mathematical representation. Further, the cognitive obstacle a key quantity was quantified by the modeler is analogous to Niss (2017) cognitive obstacle "not identifying a key real-world object to impose onto the scenario". Our theoretical perspective offers more nuance. It is not that Niali just "didn't think" to impose the quantify *amount of buffering agent in the tank,* onto the scenario. It is that Niali thought that he couldn't. Later in the interview, when the interviewer suggested that Niali use the symbol $B(t)$ to represent *amount of buffering agent in the tank* he said "We're going to solve for $B(t)$ or something, or is that something we can do?" We infer from this question that Niali anticipated solving for $B(t)$ and so he did not think he could use $B(t)$ in his equation. This comports with the findings that it is specifically skills for structuring and mathematizing that produce cognitive obstacles not mathematical nor real-world knowledge (Niss, 2017; Jankvist & Niss, 2020). However, we did not find cognitive obstacles analogous to the other two cognitive obstacles by Niss (i.e., not identifying a mathematical object and imposing unhelpful mathematical objects). Our framing fails to pick up on those kinds of cognitive obstacles because our framing has no analogue for mathematical object. Further research is needed to determine if this is a shortcoming in the quantitative reasoning-symbolic form framing, if this kind of cognitive obstacle can be detected when looking at cognitive obstacles associated with other parts of mathematizing, specifically when externalizing the structure of the scenario via symbolic forms, or even whether they could be replicated in this student population.

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