Student Reasoning in Quantum Mechanics Examined Through Modeling and Sensemaking

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As part of an effort to examine student mathematical and physical reasoning in quantum mechanics, a paired, semi-structured, think-aloud interview was conducted. The students were asked to interpret a quantum mechanical operator expression in a functional (position) representation and to think about an analogous expression in Dirac notation. When the students ceased to make progress on this task, they were given a modeling task in which they constructed an eigenvalue equation for a different operator in quantum mechanics. After examining their newly constructed eigenvalue equation, students were able to determine more precisely the nature of the original expression given in a functional representation. These data were coded in accordance with mathematical modeling and mathematical sensemaking frameworks to examine the intersection of the two frameworks.

Keywords: Quantum Mechanics, Modeling, Sensemaking

Introduction

Due to its prevalence in modern research and applications in industry, quantum mechanics is an essential part of an undergraduate physics curriculum. Research on student understanding of quantum mechanics can be largely differentiated by the instructional methods used with the student populations. Undergraduate quantum mechanics is typically taught with either a functions-first or spins-first approach. In a functions-first approach, instruction first focuses on continuous systems represented in functional notations. This often requires solving differential equations to model (e.g, Griffiths, 2018). In spins-first instruction, students begin with discrete systems represented in Dirac\bra-ket notation (a vector-like notation developed for use in quantum mechanics) before moving onto continuous systems in functional notations (e.g., McIntyre et al., 2012).

One motivation for spins-first quantum mechanics courses is that they tend to fall more in line with graduate instruction. They allow for students to build intuition for and a qualitative understanding of quantum mechanical systems while working with two-state systems, described by relatively simple mathematics, before moving onto the continuous systems that require functional representations (McIntyre et al., 2012). Ideally, students would be able to apply the ideas learned in the context of spins to continuous systems. If this were the case, one could reasonably expect to see evidence of mathematical sensemaking during this transition. Sensemaking in physics is also closely related to modeling frameworks and activities in mathematics education research. The interplay between modeling and sensemaking will be discussed herein. We begin to address these issues by examining students' reasoning and sensemaking while working with multiple representations in the context of quantum mechanics.

Background

We first situate this study among the relevant literature associated with teaching and learning in quantum mechanics. This is followed by a more in-depth discussion of modeling and mathematical sensemaking, which are the two primary frameworks used in analysis.

Teaching and Learning in Quantum Mechanics

Much of the work on student understanding of upper-division quantum mechanics has focused on identifying difficulties students face in engaging with the material. Work has been done at varying levels of specificity ranging from specific concepts, contexts, or calculations to broad overviews of difficulties. Some of the specific concepts include time dependence (Emigh et al., 2015) and measurement (Gire & Manogue, 2008). Others have focused on students' use and understanding of notations (Gire & Price, 2015; Kohnle & Passante, 2017; Schermerhorn et al., 2019). Many of these have been used as the basis for curriculum development projects aimed directly at the identified difficulties (DeVore & Singh, 2015; Emigh et al., 2018; Kohnle & Passante, 2017; Singh, 2016). Others have examined student understanding from the perspective of resources in efforts to determine what may lead to certain lines of unproductive reasoning (Gire & Manogue, 2008, 2011). A variety of studies have addressed how students think about operator and eigenvalue equations in quantum mechanics (Her & Loverude, 2020; Singh, 2008; Singh & Marshman, 2015; Wawro et al., 2020). There is also a growing body of research around representations in quantum mechanics due to their variety and the need for both fluency with and fluidity between them. The three primary representations in quantum mechanics are Dirac, matrix-vector, and functional. The first two are closely related and well suited to modeling discrete systems, while functional representations are well suited to modeling continuous systems. The *resources framework* posits that people have a variety of cognitive resources that are activated in different contexts and utilized in reasoning (Hammer, 2000). Gire and Manogue identified student resources *quantum measurement as agent* and *operator as agent* that could be combined unproductively to lead students to the idea that an operator acting on a state is a representation of making a measurement of the corresponding observable (Gire and Manogue, 2008). This association between operators and measurements is consistent with other findings in the mathematics education community (Wawro et al., 2020).

Mathematical Modeling Frameworks

Modeling activities are commonly implemented in mathematics instruction to present students with opportunities to practice developing their skills in a real-world context, or to demonstrate how different mathematical concepts can manifest in the real world. This has led the mathematics education community to develop several different frameworks for modeling that typically depict cycles (e.g., Blum & Leiß, 2007). Given that modeling tasks often utilize contexts relevant to the natural sciences, the development of frameworks has not been exclusive to mathematics education (e.g., Modir et al., 2017; Redish & Kuo, 2015; Uhden et al., 2012).

Zbiek and Conner (2006) proposed a model consisting of a variety of processes and sub-processes both in the physical context ("real world") and the mathematical context (see Fig. 1). Their distinction between a mathematical entity and real world situation also proved productive for the analysis presented below. Some of the processes are ways one could engage with or think about a real world situation, some are ways one could engage with a mathematical entity, and others are ways that one could go back and forth between a real world situation and mathematical entity. *Exploring* is obtaining more information about the real world situation by questioning, clarifying, or paying special attention to a specific portion of the situation. By exploring one can *observe mathematically*, using mathematical ideas to describe aspects of the situation. Observing allows the interpretations of the physical system to be informed by formal mathematics, addressing a common sentiment that the mathematics and physics concepts are sometimes inextricable. *Specifying* is identifying conditions and assumptions (C&A) relevant to the real world situation. This is also a step where one is likely to come to some conclusion about the relevant conditions of the real world situation or make simplifying assumptions that will inform the *mathematization* of the problem. *Mathematization* in this case is generating a mathematical representation of the real-world situation. *Combining* is identifying that the mathematical entity one is working with has the appropriate properties and parameters (P&P). This checks that the mathematical entity is representative of the C&A. *Analyzing* includes many activities that can be used to manipulate or interpret the mathematical entity and derive one or more new properties. Included in analyzing is formal mathematical reasoning in which one utilizes their understanding of formal mathematics to derive new P&P of the mathematical entity. A subprocess of analyzing is *associating*, which is drawing on real world knowledge not necessarily associated with the real world situation being modeled to determine something about the mathematical entity. Like the way exploring mathematically provides an avenue for addressing how the interpretation of the real world situation is informed by mathematics, associating accounts for the ways in which the mathematics done are informed by physics concepts. *Highlighting* is the selection of specific properties to assist in reasoning that allow for the drawing of a mathematical conclusion. This conclusion can become one about the real world via *interpreting*. *Examining* is determining if the real world conclusions make sense given the real world situation. This could include things like validity checking or special case analysis. *Associating* and *observing mathematically* are significant features of the Zbiek and Conner model that make it well suited to analysis of physics problem solving. In physics problem solving there often processes engaged in that are not explicitly mathematical or physical. The focus on process is a primary reason this was chosen for analysis in this project.

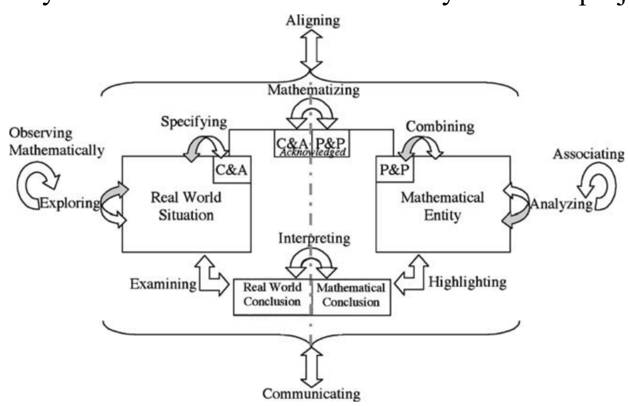
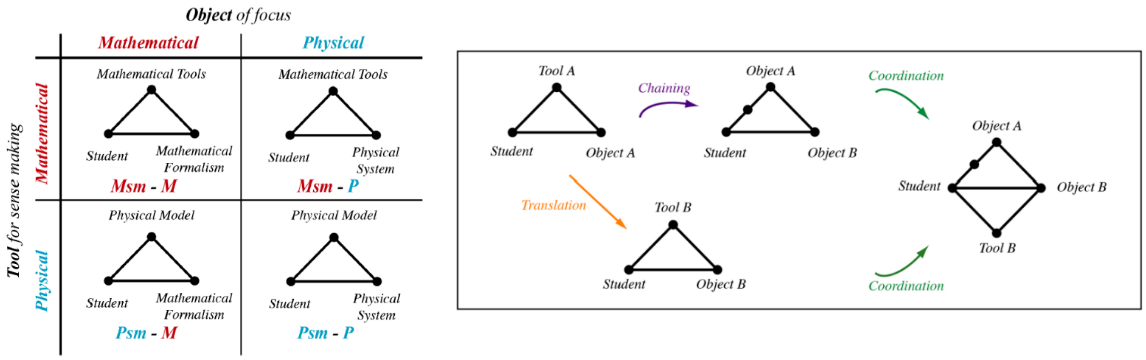


Figure 1. Framework for modeling developed by Zbiek and Conner (Zbiek & Conner, 2006).

Sensemaking

Broadly, mathematical sensemaking (MSM) can be considered a part of the larger activity of seeking coherence in which students, and even experts, seek connections between conceptual and mathematical understanding in physics (Schoenfeld, 2016). Odden and Russ worked toward building a more operational definition for sensemaking in the sciences (Odden & Russ, 2019). Their definition of sensemaking focuses on the goal – to figure something out – and a threshold, the recognition of an inconsistency. They characterize sensemaking as the work done to figure out or resolve that inconsistency.

With this definition in mind, Gifford and Finklestein developed a categorical framework for analyzing different modes of sensemaking, combining mediated cognition and activity theory (Gifford & Finkelstein, 2020). Mediated cognition is a model for cognition in which a mediator, or tool, is used to help the reasoner's understanding of an object and potentially produce some conclusion (Redish & Bing, 2009)(Vygotsky, 1978). The structure of the model is a result of activity theory, which positions the reasoner (as the subject), the tool being used, and the object being reasoned about amidst a larger context, resulting in an outcome (Engeström et al., 1999). Gifford and Finkelstein focus on the subject, tool, and object; these make up the basis for the model and for the nodes of their sensemaking triangles (see Fig. 2a).



(a) (b)

Figure 2. Categories in Gifford and Finkelstein's framework for (a) mathematical sensemaking in physics and (b) mechanisms for transitions between reasoning spaces (Gifford & Finkelstein, 2020).

For the purposes of analyzing mathematical sensemaking in physics, the tools are either mathematical tools or a physical model. These can be used to engage with either mathematical formalism or a physical system as the object. The different permutations of tools, models, formalism, and system result in four different modes of sensemaking that can be used to model reasoning. The researchers modeled transitions within and between these different modes of reasoning via three different mechanisms, summarized in Figure 2b. Students can "translate" between these modes, reasoning can be "chained" in a multi-step sequence, in which the object of one mode becomes the tool in the next mode, or two different ways of reasoning can be "coordinated" to provide two ways to make sense of the same idea (object).

Methods

The data being discussed in this paper are from a clinical, think-aloud interview with two students from a senior-level, spins-first quantum mechanics course at a mid-sized, research-focused institution in the northeast United States. The first portion of the interview was focused on how the students interpreted, both mathematically and physically, an equation for the momentum operator acting on an energy eigenstate of the infinite square well potential:

$$\hat{p}\sqrt{\tfrac{2}{L}}\sin\left(\frac{3\pi x}{L}\right) = -i\hbar\frac{3\pi}{L}\sqrt{\tfrac{2}{L}}\cos\left(\frac{3\pi x}{L}\right). \quad (1)$$

In a functional position representation, the momentum operator is expressed as a derivative with respect to position $\left(-i\hbar\frac{d}{dx}\right)$, resulting in the additional constants and change in function, making it explicitly not an eigenvalue equation. While this operation is something that students could need to calculate in solving some problems in quantum mechanics, it is not one they are likely to have thought about in the context of eigentheory. When given to students in a previous survey however, it was common for them to label it as an eigenvalue equation, making it of interest and posing the question of the ways in which students determine whether a functional equation was an eigenvalue equation.

The students were then asked to generate an analog of Eq. 1 in Dirac notation. A correct response to this could take any form in which the kets on the left- and right-hand sides of the equation are different (e.g., $\hat{p}|\varphi\rangle = |\xi\rangle$). The students engaged with Eq. 1 until the interviewer deemed that no more progress could be made. In the next portion of the interview, students engaged in a modeling task involving a system consisting of a single particle constrained to exist in one dimension. They were asked to generate an eigenvalue equation for an operator representing the position of the system (e.g., $\hat{x}|x_n\rangle = x_n|x_n\rangle$), identify the different terms in their expression, explain their individual meaning, connections that exist between the terms, and connections between the terms and the physical system. After talking through the eigenvalue equation generated by the students, the original expression (Eq. 1) was revisited, and the students were asked if anything had changed about their interpretation of the equation.

The authors engaged in collaborative qualitative analysis (Richards & Hemphill, 2018) to code student activity into the processes and subprocesses described by Zbiek and Conner. Criteria included which entity students were primarily working with (real world situation or mathematical entity) and whether they were making identifiable conclusions; certain conclusions were associated with specific processes. In the process of interpreting the given expression, students engaged in mathematical sensemaking. The framework for mathematical sensemaking in physics (Gifford & Finkelstein, 2020) was used to analyze student activity when the data reflected students' recognition of an inconsistency. Similar to the modeling analysis, this involved identifying an object and tool utilized in sensemaking. Our aim in using both frameworks is to describe the processes the students go through in reasoning with the given equation more richly than with either framework alone, and to apply that description to student understanding of eigenvalue equations.

Episodes and Outcomes

In the interest of space, we are limiting our discussion to the episodes in which sensemaking occurred. Both frameworks were utilized in the analysis. These sections provide some insight into the interplay of modeling and sensemaking, which is addressed later, in the discussion.

Recognizing Inconsistencies: A Start to Sensemaking

After some initial analysis, the students began trying to rewrite Eq. 1. The explicit manipulation and rewriting of the equation were coded as *analyzing*. The result of their re-writing was, $\hat{p}\varphi_{E_3}(x) = p_{E_3}$, which is incorrect for two reasons: it ignores the fact that there is a function on the RHS of Eq. 1, and it also implies that the energy eigenstate in Eq. 1 is also an eigenstate of the momentum operator, which it is not. The pair continued working with this expression for some time, tweaking different aspects of it before pausing long enough for the interviewer to ask what was causing them to pause. Bob then articulated the inconsistency that led to their sensemaking saying, "Yeah. And my hesitation comes from this, where we say, okay, the thing is we don't have like the, the ket [motions to RHS of $\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$]." The students had previously written the equation they referred to here. The equation is the eigenvalue equation for the Hamiltonian operator which represents energy of a system. It is a very well-known eigenvalue equation which is used repeatedly throughout the course, and so it is likely that the students are attempting to use it as a template in their reasoning.

Bob articulated that while the energy state appears on both sides of the energy eigenvalue equation, their expression did not have a state represented on the RHS. It could be argued that the students noticed that the form of the two equations did not match. If this is the case, then the use of another mathematical entity for the purposes of extracting properties and parameters of eigenvalue equations would be considered *combining*. Bob's leveraging his interpretation of the energy eigenvalue equation as a tool for making sense of Eq. 1 can be modeled as a sensemaking triangle (see Fig. 3). Their sensemaking attempt is unsuccessful since the students are not able to resolve the inconsistency between the forms of the two equations. In their reasoning, the students discuss whether the ket on the RHS of the energy eigenvalue equation is there as a formality.

> *Alice*: Well, you, I think you technically still do, but it, it just, it doesn't, this [motions to $\widehat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$,] is just a representation of it. Right?
>
> *Bob*: A representation of what?
>
> *Alice*: Like when you actually mathematically do this [pointing to $\widehat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$,] and you get your value for energy, like the ket, you wrote there. ... And that's really what you find, but when you write it mathematically and like, to be formal. You include that [points to ket on RHS of $\widehat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$,]. Which might be like not the best way to think about it ...

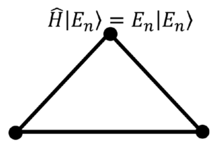


Figure 3. Model of students unsuccessful sensemaking.

Tying Modeling into Sensemaking

After the students' articulation of having essentially hit a wall, the interviewer had them move onto the modeling task, where they immediately wrote, $\hat{x}|\ \rangle = x|\ \rangle$. When asked to explain the terms in their expression, the students identified a position operator ($\hat{x}$), a measured value of position (x), and a state ($|\ \rangle$). As their discussion continued, the students made connections between elements of the expression and the real world situation. Similar to above, this was coded as *combining*, as the students were connecting their generated equation to features of the physical system. In their discussion of what label they would like to use for the kets in the expression, the students again compared it to the energy eigenvalue equation, which eventually leads them to both accept the equation $\hat{x}|L_n\rangle = x_n|L_n\rangle$ as an appropriate eigenvalue equation for an operator representing the position of a particle constrained to exist in one dimension.

The interviewer asked the students to think about what makes the equation they generated for a position operator an eigenvalue equation. Alice and Bob focused on the fact that when operating on the state with the position operator, the state does not change.

> *Bob*: … when we operate our operator, which we designate as *x* hat [$\hat{x}$], um, to represent the position on one of our states, which that could be anything, but since when we operate $\hat{x}$ on this state, we get a scalar value times the same state back. It means that we didn't, we didn't, we, it, […] - if we had changed our state, they wouldn't commute.
>
> *Alice*: By measuring, we don't change the state.
>
> *Bob*: Yeah. But by measuring, we don't change the state. We just find the scalar value that represents the state, I guess.

The students' justifications that their equation is indeed an eigenvalue equation was also coded as *combining*: they are ensuring that the mathematical entity they generated has the appropriate properties to describe the real world situation in the prompt. The primary property they reference in this justification is that the state does not change. When asked if their analysis had any impact on their interpretation, the students were able to fully articulate that the state is changed in Eq. 1 and therefore Eq. 1 is not an eigenvalue equation. The students used their general understanding of eigenvalue equations in quantum mechanics as a tool to determine the relevant features of the eigenvalue equation they generated. This reasoning was then chained to make sense of another object, the non-eigenvalue operator equation, resulting in the final conclusion. Figure 4 models the sensemaking process in its entirety including the students' final conclusion.

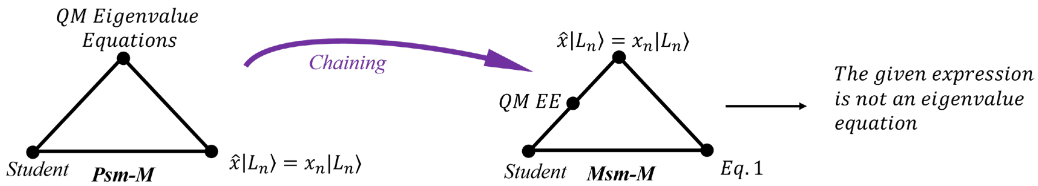


Figure 4. A model of the students sensemaking leading to their final conclusion about Eq. 1.

Discussion

In this study, Zbiek and Conner (2006)'s modeling framework is used to get a fuller picture of the processes students engage in as they work through the tasks, while Gifford and Finkelstein (2020)'s sensemaking framework is utilized to deepen the understanding of how the students resolve an inconsistency in their understanding while working in a modeling frame. When the students recognize an inconsistency in their analysis of Eq. 1, they shift into a sensemaking frame in order to resolve that inconsistency. That is not to say that a modeling frame is requisite for sensemaking, only that the two can coexist, as they do in this case study.

Odden and Russ (Odden & Russ, 2019) claim that sensemaking is often a part of modeling activities, and could potentially occur in parallel with modeling. Our study documents an example of sensemaking within modeling, and explicitly identifies behaviors and strategies students use to resolve an inconsistency. The modeling and sensemaking frameworks come together here to provide insight into the kinds of activities students engage in when given a task that places them in what is typically the middle of a modeling cycle, how they go about trying to resolve an inconsistency, and how analogous modeling tasks can help students refine their reasoning on a task with which they are struggling.

While productive, the Zbiek and Conner (2006) model may not be sufficient to model more advanced physics problem solving. In coding, the authors had several discussions in which they were unable to singularly code some of the students' work because it was difficult to determine whether the object of the students' thinking was a mathematical entity or physical system (e.g., real world situation). We suspect this is due to the blended nature of quantum mechanics content rather than a lack of student articulation thereof. This may suggest a need for modifications to existing frameworks, or a new framework entirely, that accounts for the ways mathematics and physics are blended in physics reasoning and modeling.

The primary limitation of this study is that the data are from a single pair of students so results may not be generalizable. However, these students were articulate and thoughtful, and the ideas they present corroborate previous findings in the literature. We look forward to further empirical and theoretical work in this and other physics topics to provide additional insight.

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