Using CAS to Promote Students' Ways of Thinking Through Observation and Conjectures: The Case of Eigenvalues and Eigenvectors

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Linear algebra is an important topic in mathematics and many other disciplines. In this paper, we consider a set of digital interactive figures (I-figs) using Mathematica created for linear algebra students in an introductory course. The figures were designed to facilitate students' ability to visualize and work with eigenvalues and eigenvectors while minimizing computation for the benefit of conceptual focus while looking at an unlimited number of examples. The worksheets provide a foundation for motivating students to participate in a system of observation, conjecture, proof, and theorem. Based on our preliminary analysis of students' reflections on the I-figs, we found that students were confident in making and believing example-based conjectures.

Keywords: eigenvalues and eigenvectors, technology, observation, and conjecture

Literature Review

Already an important topic in mathematics, science, and engineering, the relevance of linear algebra has expanded into other areas as it provides the foundation for a wide range of data-driven and AI-related techniques. Linear algebra plays a central role in almost any area that uses quantitative information and methods to process data, hence the way linear algebra is taught is of paramount importance.

In the early 1990s, the Linear Algebra Curriculum Study Group (LACSG) recommended that "faculty should be encouraged to utilize technology in the first linear algebra course" (Carlson et al., 1993). As a part of this initiative, David Lay implemented the use of computer algebra systems (CAS) educational support such as MATLAB, Maple, and Mathematica in his book *Linear Algebra and its Application* (Lay, 1994). For example, in working with matrices and vectors, solving systems, matrix multiplications, finding inverses, and reducing a matrix to a reduced echelon form, technology can help with accurate and fast calculations as well as allow one to concentrate on the deeper conceptual aspects of the discipline that are often difficult for students to grasp. In addition, technology can be employed to build a visual image of the details, as well as the big picture, and allows for predictions, investigations, and producing conjectures.

Despite national initiatives in the US on teaching linear algebra with MATLAB, for example, the ATLAST project initiated by Steven Leon and his colleagues (Leon, Herman, & Faulkenberry, 2002) in the early 90s, systematic studies of linear algebra instructors' thought processes and their students' feedback on the effect of technology on their understanding are lacking. In a survey paper by Stewart, Andrews-Larson, and Zandieh (2019), the authors looked at linear algebra education papers from 2008-2017 in mathematics education journals and found almost no systematic classroom studies on the effectiveness of the use of Mathematica, Maple, Python, or MATLAB in the classroom, with the exception of a study of using MATLAB in mathematical modeling (Dominiques-Garcia, Garcia-Plana, & Taberna, 2016). However, the survey paper found studies on eigentheory using Geometer's Sketchpad (e.g., Gol Tabaghi, 2014; Caglayan, 2015), and Geogebra (Beltran-Meneu, Murillo-Arcila, & Albarracin, 2016) focused on the geometric aspects of the topic. The linear algebra recommendations by LACSG

2.0 also emphasized the importance of technology in teaching linear algebra (Stewart et al., 2022).

The second author has developed a series of digital interactive figures (I-figs) that aim to provide students with an opportunity to play with linear algebraic objects in an embodied format meant to be easy-to-access and requiring minimal time investment. Linear algebra students' conceptual understanding of eigenvalues and eigenvectors have been a topic of study in some studies (e.g., Thomas & Stewart, 2011; Salgada & Trigueros, 2015; Wawro, Watson, & Zandieh, 2019), largely focused on student thought processes. We also wish to study student ways of thinking, with the added structure of a series of I-figs targeting properties of eigenvalues and eigenvectors.

Theoretical Perspective

The theoretical framework for this study will utilize Harel's (2008) ways of thinking and Tall's three worlds of mathematical thinking (2010). Harel (2008) introduced the notion of a mental act as actions such as interpreting, conjecturing, justifying, and problem solving, which are not necessarily unique to mathematics. These several types of mental acts are the foundation of ways of thinking defined as "a cognitive characteristic of a mental act" (p. 269). These ways of thinking have the characteristics of being able to abstract, generalize, structure, visualize, and reason logically. Tall (2010) describes the embodied world as "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns… and other forms of figures and diagrams" (2010, p. 22). In his view, "The world of operational symbolism involves practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as a number)" (2010, p. 22). The formal world "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (2010, p. 22).

Our research questions are: What are the effects of the use of these I-figs on students' ways of thinking involving a system of observation and conjecture and how does that influence their understanding of proof and theorem? How did the use of the I-figs influence learning of eigentheory?

Methods

In this case study, 16 participating students were enrolled in an honors section of introductory linear algebra at a large public land-grant R1 university in the USA. These students were first and second-year students, with most declared or interested in degrees in mathematics, science and engineering, or medicine.

Five I-figs related to properties of eigenvalues were collected as one digital worksheet for students to explore. The worksheet was given to the students over a weekend's time to complete, immediately following the first class on eigenvalues and eigenvectors. In this lesson, students were shown the definition of eigenvalue and eigenvector with necessary conditions such as non-zero eigenvectors and uniqueness of eigenvalues for an eigenvector. By the time of worksheet participation, students had not practiced or discussed in class how to compute eigenvalues or eigenvectors, nor had any of the eigenvalue properties displayed in the worksheet been demonstrated. Students were instructed to spend approximately 15 minutes manipulating the digital figures and look for any observable patterns or trends.

Three forms of written data were collected at different times following the students' participation. Immediately after their time with the worksheet, students were asked to write a few paragraphs describing anything that they found interesting or noteworthy while working in the digital worksheet. Second, a few days later in the following in-person class time, students completed a short quiz asking if they could recall the definition of eigenvalue and eigenvector, and again to recall anything that stood out to them about the worksheet. Lastly, two weeks after the worksheet was administered when the class had concluded their study of eigenvalues and eigenvectors, students were asked to complete a final exit survey consisting of seven short answer response questions: (1) In what ways, if any, did you find the digital worksheet useful in understanding the concept of eigenvalues and eigenvectors? (2) What pros/cons do you see with this activity being a digital worksheet versus a pen-and-paper worksheet? (3) Describe, if any, the connections you see between the digital worksheet and the following week's content on eigenvalues and eigenvectors. (4) Do you like the idea of these exploratory activities before the content is introduced in class? (5) Would you have preferred a different sequencing? Please share your reasoning. (6) In our modern data-driven society, how do you see the relationship between linear algebra (matrices as data) and technology? (7) What part of the digital worksheet stood out to you as most important? Why?

As another data perspective, the instructor of the course (the first author) logged teaching reflections after each class meeting. These reflections included a summary of the day's class content, delivery, and notes about what did and did not go well, especially including important (from the instructor's perspective) student interactions. The instructor had weekly meetings with the rest of the research team and discussed their perspectives about the class further. The research team helped to design the study and research tools. Open coding by Strauss and Corbin (1998) was performed to analyze the data. In this paper, we will only analyze students' responses to their immediate post-worksheet reflection.

Preliminary Analysis

Some of the common themes in students' data so far have been as follows: play and exploration, pattern/non-pattern seeking, desire for relevance, and mathematical (formal) conjecture. These reflections ranged from one to three paragraphs, typically no more than half a page of writing.

We will begin by analyzing student responses relevant to each I-fig. The first I-fig (Figure 1) treats eigenvectors geometrically. Students may adjust the second component of v manually or use presets to snap to two eigenvector relations. Only four of the thirteen respondents addressed this geometric figure in their reflection. Those that did unilaterally employed physically embodied language surrounding their work, such as "It was interesting to scale the vector with my hand and check the gradual change of the values on my own" (S09). However, only one student explicitly named those special vector equations as demonstrating an eigenpair relationship.

For the second and third I-figs (Figures 2), eight students were able to correctly comment on the intended patterns in these I-figs (the same students for each). From this activity, these students make assertive statements (conjectures) such as "when you scale a matrix, the eigenvalues are scaled the same amount" (S03) or "When the matrices were put to a specific power, the eigenvalues were also just put to that power" (S05). From these notes we observe that those eight students all communicated a confidence that their perceived relationship is true (without supplying or referencing any justification), and also were able to supply a productive phrasing of the appropriate eigenvalue property, without having seen or otherwise been exposed to those properties in the class. As mathematics instructors, we can capture this as students' displaying an emergent way of thinking of the form (way of thinking: I can build a conjecture with a finite set of examples) and (way of thinking: I feel strongly that my conjecture is true). Notably, in the second way of thinking, students are unbothered by the absence of any supporting proof and have yet to become skeptical that a small set of examples can be used to identify a general trend.

Only one student who addressed these two I-figs proposed related though fundamentally incorrect statements such as "when scaling a eigenvalue, it affects the matrix" (S07). Nevertheless, we see the fundamental idea (of matrices and their eigenvalues sharing a scalar relationship) communicated, though this student's implication is reversed.

Next, Figure 3 shows an I-fig meant to display eigenvalue properties (or lack thereof) in relation to matrix operations on two matrices A and B. Seven students remarked upon this I-fig. Of those, five students were able to either state that the eigenvalues of A and B do not simply determine the eigenvalues of AB or A+B ("adding two matrices together does not necessarily add the matrices' eigenvalues, nor does multiplying two matrices together multiply the eigenvalues" (S06)), or stated that they were not able to find those patterns ("I wasn't sure how there is a correlation between eigenvalues of matrixes and their sums and products" (S12)). We note that these students leave open the possibility of there existing some pattern or rule that they simply were unable to find.

The final I-fig (Figure 4) concerns the relationship between eigenvalues, trace, and determinant of a matrix. Six of the thirteen respondents were able to conjecture a relationship, with five of those being correct ("I realized that adding the eigenvalues gives you the trace, while multiplying the eigenvalues gives you the determinant" (S07)). The final student observed "With the matrix provided, if I took the sum of all the eigenvalues, I received the trace of the matrix. Then, if I took that sum and multiplied by negative two, I got the determinant of the matrix" (S12). We see this student extending a previously identified way of thinking (I can build a conjecture with a finite set of examples) to now creating an incorrect conjecture apparently from a single viewed example. Again, showing no hesitation in endorsing this conjecture, indicating a lack of a way of thinking such as a mathematical conjecture requires a proof to be accepted.

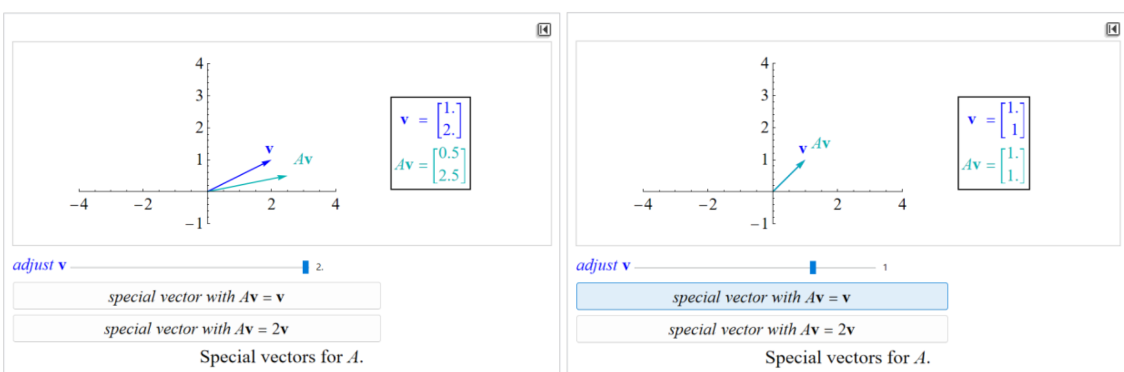

Figure 1. Geometric and algebraic connection of eigenvalues and eigenvectors.

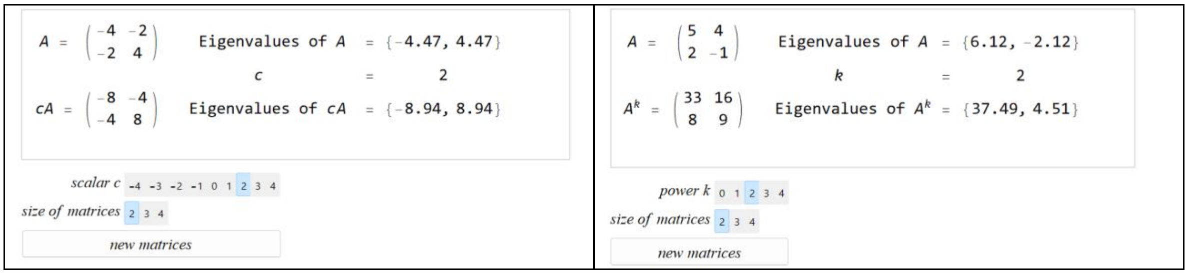

Figure 2. Effect of scalar multiplication and matrix exponentiation on eigenvalues.

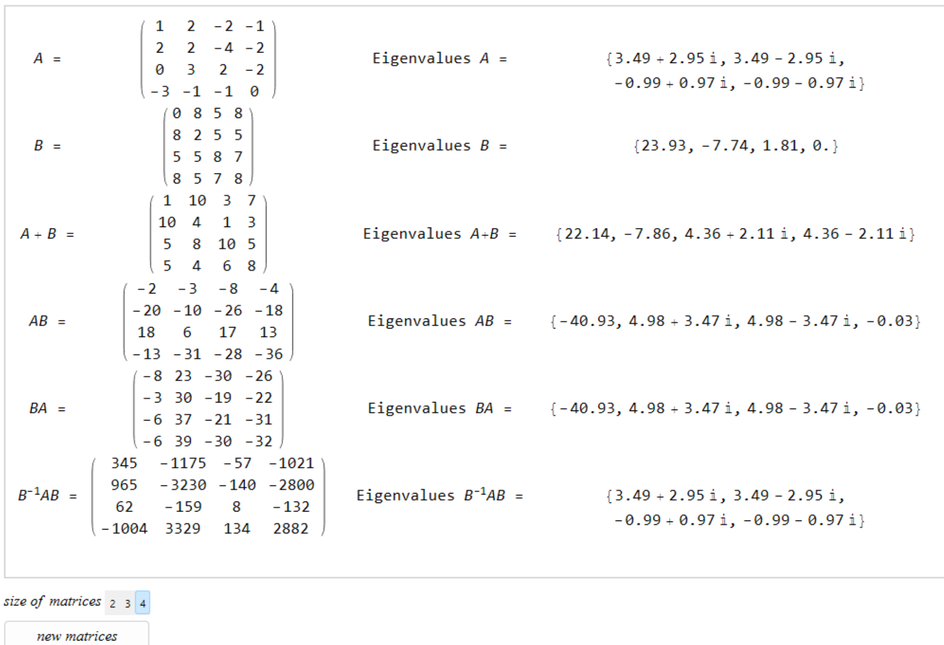

Figure 3. Eigenvalues and sums and products of matrices.

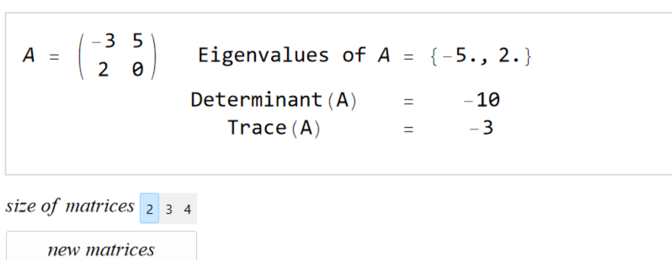

Figure 4. Relationship between eigenvalues, determinant, and trace.

Concluding Remarks

As we continue working on this research study, we plan to develop the theoretical framework further, complete the coding, and analyze the data by employing our theoretical framework hybridizing ideas of Tall's (2010) three worlds of mathematics and Harel's (2008) ways of thinking and understanding. Our analysis will also include some recommendations for teaching linear algebra using and developing CAS worksheets, and to target future iterations and additional worksheets.

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