A Conceptual Analysis of Expressing Distances Algebraically within the Cartesian Plane

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This theoretical report offers a conceptual analysis of expressing distances on graphs of functions in the Cartesian plane algebraically. We frame this mental activity as a connection between the algebraic and graphical registers, and describe three key connections it comprises: (1) differences express distances between two positions, (2) points are ordered pairs of distances from the axes in the Cartesian plane, and (3) equations give the relation between x and y for every ordered pair on a graph. We detail the conceptual components of each connection, which involve a component from each of the graphical and algebraic registers and an underlying interpretation that forges the connection between the two. This conceptual analysis makes explicit the complex cognitive steps involved in algebraically expressing distances on graphs of functions, with implications for researchers and practitioners using algebraic and graphical representations together.

Keywords: Graphical representations, algebraic expressions, conceptual analysis, Cartesian plane

The central goal of this theoretical report is to offer a conceptual analysis (Thompson, 2008) of the mental activity involved in representing distances between functions graphed in the Cartesian plane using algebraic expressions. Many mathematical objects involved in key results of undergraduate mathematics are often represented graphically and expressed algebraically in textbooks and curricular materials, such as the difference quotient in the limit definition of the derivative. Specifically, students' ability to use algebraic expressions to represent distances within graphs may support them in making sense of such representations that are paired together. Further, this ability may also assist students in conceptualizing other key results, such as how integrals find area under and between curves, and volumes of solids of revolution. In this paper, we consider the mental activity involved in expressing distances algebraically by centering our discussion around the following task and asking: what are the conceptions involved in algebraically expressing the horizontal distance between any point on $y = \sqrt{x-1}$ and a point on $x = 2$ (for all *x* such that $1 < x < 2$) in terms of *x* and in terms of *y*? (Figure 1).

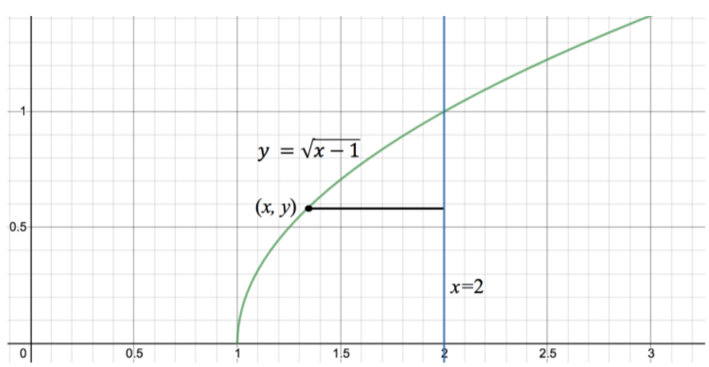

Figure 1. Graph showing a horizontal segment from a generic point (x, y) marked on $y = \sqrt{x-1}$ *to* $x = 2$*. The length of this segment may be expressed as* $2 - x$ *or equivalently as* $2 - (y^2 + 1)$*.*

Conceptual Analysis of Expressing Graphical Distances Algebraically

We engage in the process of conceptual analysis consistent with Thompson (2008) as describing "what students might understand when they know a particular idea in various ways" (p. 42) to detail the mental steps involved for students to conceptualize an algebraic difference expression as representing a horizontal (or vertical) distance within a graph in the Cartesian plane. Based on previous research and work with students on such tasks (e.g., Parr et al., 2021), we theorize that there are three main connections between the graphical register and algebraic register that underly this conception: (1) differences express distances between two positions, (2) points are ordered pairs of distances from the axes in the Cartesian plane and (3) equations give the relation between x and y for every ordered pair represented at a point on a graph. These three connections build upon each other to comprise the connection among *distances between functions in graphs* and *(algebraic) difference expressions*. We describe each of these connections between the graphical and algebraic register, as well as the underlying mechanism for how the connection is forged. We see potential obstacles for students within conceptualizing each component of the graphical register, algebraic register, as well as the mechanism of the connection. We will describe each of the three steps and previous research highlighting potential obstacles for students within each.

Connection 1: A difference represents a distance between two positions

From our perspective, the first main cognitive step to expressing distances on graphs of functions involves using a difference expression to represent a distance between two positions in a single dimension. Within the graphical register, this connection involves conceptualizing distance as a quantity, that is, a measurable attribute (Thompson, 1990, 2011) of the spatial arrangement of two positions within a number line or graph. Within the algebraic register, this connection involves operating from a "*determine the difference*," (Selter et al., 2012) also referred to as a "*comparison*" (Usiskin, 2008) model of subtraction, rather than a *takeaway* model. Connecting these conceptualized distances with difference operations relies on a magnitude interpretation of symbols (Parr, 2021). We summarize these components that result in this connection between differences and distances between two positions in Figure 2.

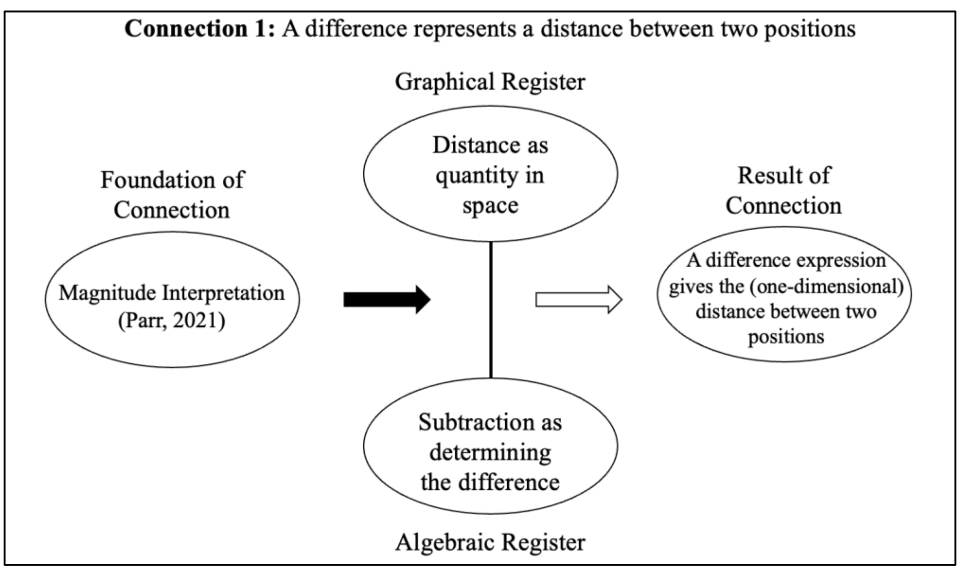

Figure 2. The components of connecting distances in graphs with difference expressions.

Previous research suggests that students may not use these ways of reasoning underlying the connection involved in representing distances with differences. Within the graphical register, conceptualizing a distance as a quantity in space is a non-trivial component of this connection. The Cartesian coordinate system relies on distances to relate points, yet students may not conceive of distance between points as relevant when viewing positions on a number line or graph. Alternatively, they may conceive of positions as labeled at arbitrary locations, or use some other comparison, such as relative location, without explicitly attending to or conceptualizing measurable distance between positions (Parr, 2021). At the elementary level, students may not necessarily place non-consecutive numerical values at appropriate distances apart from one another on a linear-scaled number line (Saxe et al., 2013). Instead of using a number line as a measurement model, some students may use it as a counting model (Diezmann & Lowrie, 2006), counting the number of tick marks or intervals (Mitchell & Horne, 2008). Within the algebraic register, conceiving of subtraction as an operation that determines the difference between two amounts that are compared is not a given for students. Previous research has found that the comparison operations are those that pose the most difficulty for students among types of subtraction problems in early grades (Stern, 1993). When asked to interpret an expression involving a difference, students may think exclusively of a counting down or takeaway model, rather than a comparison one (Figure 3).

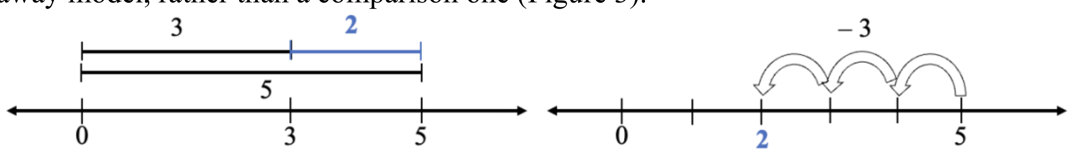

Figure 3. A determine the difference (comparison) model of 5-3 (left) vs. a takeaway model of 5-3 (right).

Students may also default to using the takeaway model for subtraction on a number line, and may not as readily use a determine the difference model, perhaps because the former is more commonly encountered in school mathematics and everyday situations (e.g., Selter et al., 2012). Even at the undergraduate level, students may default to using a takeaway model of subtraction. This was the case with a student, Peter who was at first unable to reconcile a difference expression in $|x-1| < \delta$ with a graph of a function showing a shaded vertical strip centered at $x = 1$ to represent all values of x within a given distance (δ) of 1. (David, 2018).

We view the magnitude interpretation (Parr, 2021) of a symbol (either a number or variable) or expression as the foundation of the connection between the graphical representation of distance and the algebraic operation of subtraction to determine the difference. A magnitude interpretation of a symbol recognizes that a symbol refers to both a position as well as a measure of a distance from 0 (Parr, 2021). To represent the "determining the difference" model of subtraction on a number line, one employs what Parr et al. (2021) refer to as a *composed* magnitude interpretation. For example, to understand conceptually why the difference expression $x_2 - x_1$ yields the distance between two positions, x_1 and x_2, in one-dimensional Cartesian space, one must first understand the positions x_1 and x_2 as distances from the origin themselves. Then, one can apply the operation of subtraction to find the difference between the length of x_2 and the length of x_1 to express the distance between position x_1 and x_2 as $x_2 - x_1$, as shown in Figure 4. Students may not readily use a composed magnitude interpretation of difference expressions, even in situations where it would support further mathematical activity. At the undergraduate level, students may still use a cardinal interpretation, counting tick marks or spaces to interpret a difference expression on an axis, rather than as a distance between two points, as in the case of Annie and Kate reported in Parr (2021).

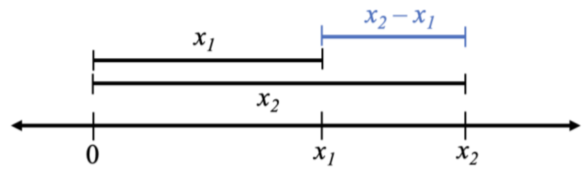

Figure 4. *Composed magnitude interpretation of the expression $x_2 - x_1$ involves conceiving of x_1 and x_2 themselves as magnitudes, or distances measured from 0.*

Connection 2: A point given by (x, y) represents an ordered pair of distances from axes in the Cartesian plane

The next connection involved in expressing distances in the Cartesian plane is the connection that an ordered pair of values, (x, y), that locate a point in the Cartesian plane represents an ordered pair of distances measured from the point to each axis. This connection builds on the prior one, such that x is a magnitude from the origin horizontally and y is a magnitude from the origin vertically. What is new in this step is the combination of the two magnitudes into two-dimensional space, so that a single entity given by an ordered pair is connected to the pair of distances. The foundation of this connection is value-thinking (David et al., 2019), in which one envisions a point as a multiplicative object (Saldanha & Thompson, 1998), at once a single entity that is comprised of two components conceived of simultaneously. When combined with a magnitude interpretation from the previous connection, one envisions a point as a multiplicative object of a pair of distances to the axes (Figure 5).

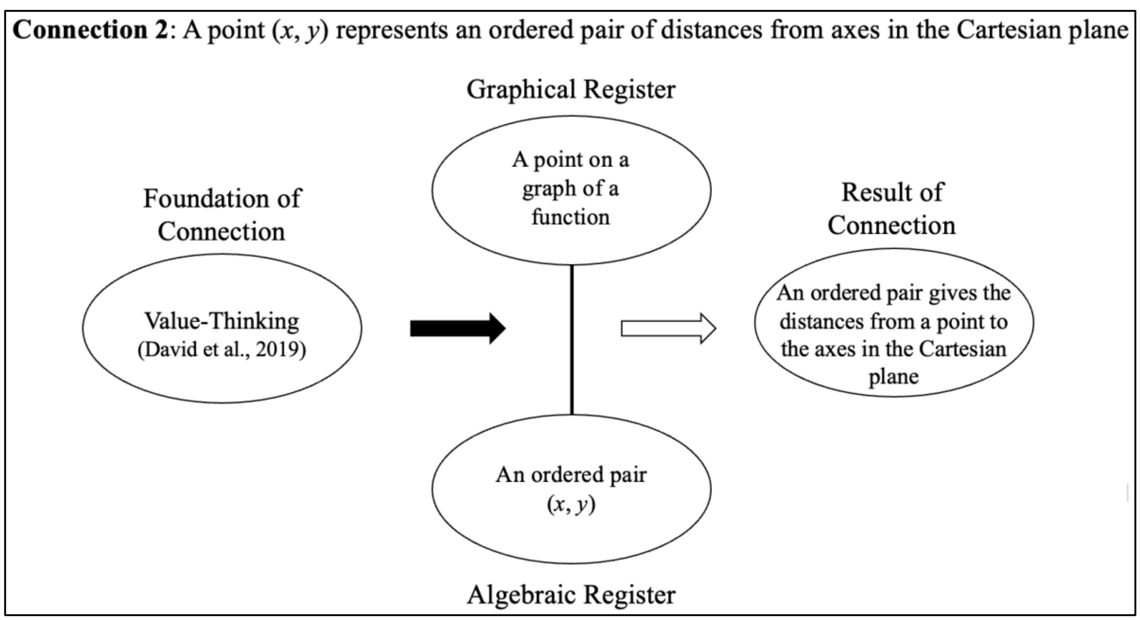

Figure 5. *The components of connecting points on graphs with ordered pairs of variables.*

Prior research suggests that students, both at the secondary and undergraduate level, may not connect a point on a graph of a function with an ordered pair of input and output values of the function via value-thinking. Students may instead associate a point solely with a single value- often the output of a function, rather than a pair of values, as in the case of location-thinking

(David et al., 2019). When students do connect a point with an ordered pair, they may conceive of the ordered pair purely as a directive for how to locate a point (Thompson et al., 2017)- right (or left) some amount *x* and then up (or down) some other amount *y*. Students who view ordered pairs this way may not be able to unite pairs of magnitudes represented on axes in a single point, even after appropriately plotting points themselves using the over and up technique (Frank, 2016). Thus, we view building the connection between ordered pairs and pairs of distances from axes as an essential part of supporting students in expressing distances between points on functions of graphs.

Connection 3: An algebraic relationship of *x* and *y* can be used to find equivalent expressions of distances

The third connection between the algebraic and graphical register that we view as essential to expressing distances is the Cartesian connection (Moschkovich et al., 1993). The Cartesian connection states that the set of all ordered pairs of points on a graph of an equation correspond to the complete set of pairs of values satisfying the equation. Using this connection with the prior two, in which ordered pairs for points give pairs of distances to the axes, allows one to conceive of the algebraic equation relating *x* and *y* as a way to express distances in the graph flexibly in terms of *x* or *y* as needed (Figure 6).

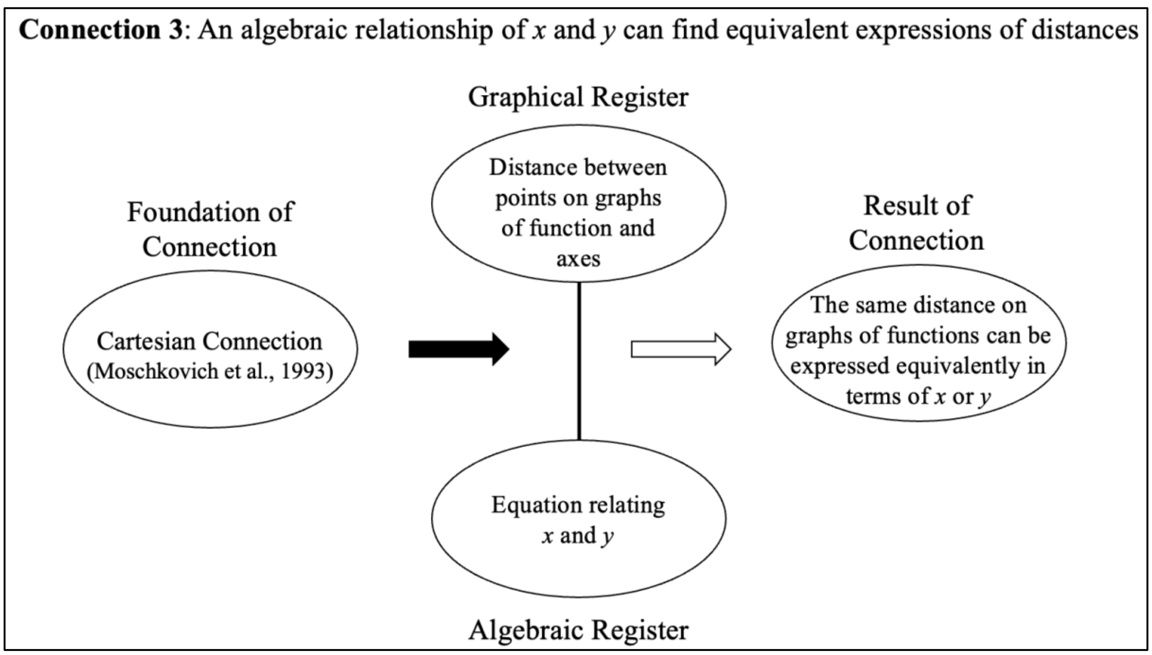

Figure 6. The components of connecting distances on graphs with equations of x and y.

Previous studies have shown that students may not recognize the Cartesian connection when working with graphs of functions and their algebraic relationships (Dufour-Janvier et al., 1987; Glen & Zazkis, 2021; Knuth, 2000; Moon, et al., 2013). Instead of viewing an equation for a linear function as comprising the set of all ordered pairs of values satisfying the equation, which can be plotted as points comprising the graph of the line, students may see a linear equation as a directive for how to draw a graph of a line using slope and intercept information (Parr et al., 2021). In the words of Todd, a Calculus student, "y =2x+1… tells you how it's [the graph is] going to look, so the slope would be 2, the y-intercept would be 1" (Parr et al., 2021, p. 219).

Representing distances on two-dimensional graphs of functions using algebraic expressions is a complex cognitive activity. In order to conceptualize why expressions such as $2–x$ and $2–(y^2+1)$ in Figure 1 represent the distance depicted, students must connect the graphical and algebraic register in the three ways described in this section: 1) a distance between can be modeled using a difference, 2) an ordered pair of values gives the distances the associated point is located from the axes in the Cartesian plane, and 3) an algebraic relationship between x and y can be used to flexibly express distances within the graph of the relationship in terms of x or y.

In Figure 7, we show how these connections combine to connect difference expressions with distances between two points on functions in the Cartesian plane. Connection 1 supports students in expressing a straight-line distance between two positions x_1 and x_2 as $x_2–x_1$. Connection 2 supports students in expressing a straight-line distance from an axis in the Cartesian plane using the coordinates of the point, in this case, the horizontal distances as x_1. Combining these two connections supports students in describing the horizontal distance between two points in the Cartesian plane as $x_2–x_1$. Assuming the two points shown are located on functions f and g, respectively, Connection 3, the Cartesian connection, would support a student in expressing this same horizontal distance in terms of y, as $g^{-1}(y_2) – f^{-1}(y_1)$.

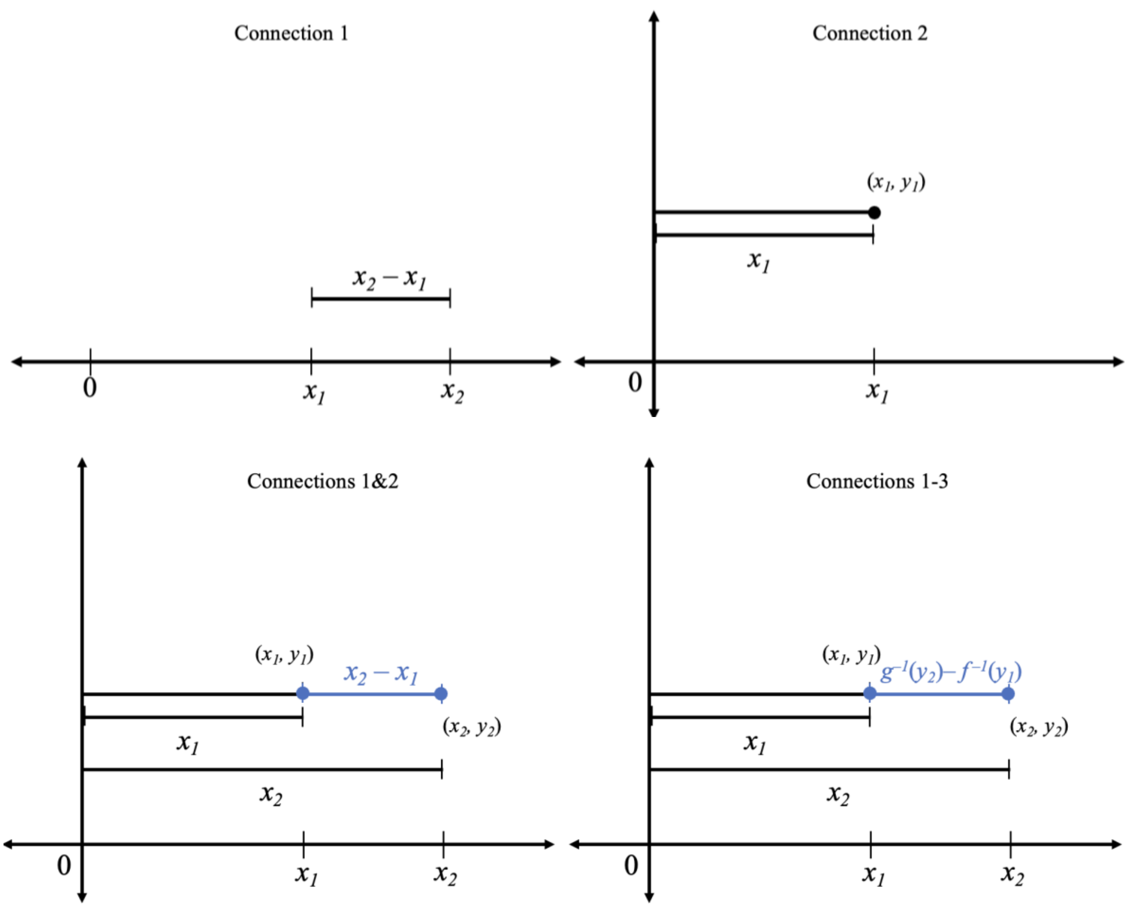

Figure 7. How Connections 1-3 combine to express horizontal distance between two points in the Cartesian plane.

Discussion

The conceptual analysis we present in this paper offers a detailed account of the cognitive connections need for students to meaningfully use an algebraic difference expression to represent distances between functions in graphs. This conceptual analysis has implications for both research and practice. First, the three connections we highlight between the algebraic and graphical register uncover the complex nature of the cognitive steps involved expressing distances algebraically. At the undergraduate level, researchers and instructors may take for granted students' fluency in moving between two representations. In fact, this connection is not included in typical calculus curricular materials and instruction. Yet, our research and this analysis suggest that students may benefit from more explicit instruction as to how these representations may be used together. In addition to drawing attention to its underlying complexity, this analysis may serve to support the development of learning trajectories and tasks to assist students in connecting graphs with algebraic expressions meaningfully. Further, this conceptual analysis may alert researchers and practitioners to potential issues in students' reasoning that they may anticipate when using subject matter represented both algebraically and graphically. These issues may arise in either conceptualizing the objects in the graphical register, the objects in the algebraic register, or the way in which these objects are connected.

More broadly, the structure of the conceptual analysis we offer here may serve as a model for others related to connections among registers. Many prior examples of conceptual analysis in the literature attend to students' understanding of mathematical ideas, but may not explicitly attend to representational structures involved in conceiving of or communicating these ideas. We add to the example of Lee et al.'s (2018) conceptual analysis of coordinate systems by attending to the representational structures involved in connecting graphical distances with algebraic expressions. We decompose the larger connection in question into sub-connections, which each include components in the algebraic register, graphical register, and an underlying interpretation. This structure may serve as a model for future conceptual analyses on other connections between the graphical and algebraic registers, or any other combinations among mathematical registers.

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