Teaching Practices Addressing Multiple Uses of a Word in Mathematics – Case for Derivative

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We investigated teaching practices aiming to communicate multiple uses of a common term in mathematics with students focusing on the use of a single word "derivative" for both objects "the derivative at a point" and "the derivative function" using commognitive approach and intellectual needs. Our analysis of a teacher's teaching explicitly attending to this feature revealed several teaching practices with this aim: discussing discursive rules for distinguishing the two uses and using another mathematical object for connecting the uses (e.g., slope); discussing a colloquial object that has a similar feature and is familiar to students, which provided a familiar discourse that students could rely on; and avoiding multiple uses of a common word in one context, Those teaching practices are impacted by multiple uses of a common signifier in communication and mathematical relations among the two objects objectified by a common word, but also impacted how the objects became related.

Keywords: commognition; teaching practice; derivative; words with multiple uses

Multiple uses of same signifiers (words or symbols) have been known to cause potential difficulties in teaching and learning of mathematics (e.g., Biza & Zachariades, 2010; Thompson & Rubenstein, 2000). For example, various calculus terms are used as both process and object or as both function and a specific value (e.g., College Board, 2023; Güçler, 2013). This study focuses on the discourse about the derivative to examine the teaching practices that aim to help students learn about multiple uses of a common term in mathematics. In introductory calculus, derivatives can be separated into the derivative at a point and the derivative function (we use this term for a function obtained by differentiating another function). The observation that the word "derivative" is included in "the derivative at a point" and "the derivative function," the word "derivative" alone is often used for these objects (e.g., College Board, 2023; Stewart, 2016), and students may face challenges distinguishing or relating these objects (Font & Contreras, 2008; Park, 2013), motivated our study investigating potential teaching practices aiming to promote students' learning of the use of this common term with the following research question:

What can be the teaching practices that aim to promote students' learning of multiple uses of a common term for two mathematical objects in formal mathematical discourse?

We adopted the commognitive view of *learning* mathematics as "the process in which students extend their discursive repertoire by individualizing the historically established discourse called mathematics" which we refer to as canonic discourse (Sfard, 2018, p. 222). Such extensions are viewed as meta-level learning when they involve changes in rules of discourse (e.g., learning how to choose one of multiple uses of a new word like the 'derivative'). In contrast, object-level learning only involves changes or expansions of properties of objects that are already introduced in the discourse (Valenta & Enge, 2022). Individualizing a discourse means developing one's ability to communicate with others and oneself according to the rules of the discourse. Given the commognitive view of teaching, "the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse" (Tabach & Nachlieli, 2016, p. 303), we assume that there are teaching practices aiming to help students learn about how to apply those rules. Our goal is to reveal such practices in the context of the derivative based on a case study of a teacher explicitly addressing this discursive feature in class.

Theoretical Background

This study builds up on the existing literature about teaching and learning of multiple uses of a signifier in mathematics (Biza et al., 2008; Lamb et al., 2012; Rubenstein & Thompson, 2001; Thompson & Rubenstein, 2000; Zazkis & Kontorovich, 2016). We adopted the commognitive view of teaching and teaching practices (Cooper & Lavie, 2021; Nachlieli & Elbaum-Cohen, 2021; Tabach & Nachlieli, 2016) and intellectual needs (Harel, 2008, 2013) to identify teaching practices aiming to help students learn about multiple uses of 'derivative' and tasks they address.

Teaching and Teaching Practice Promoting Metalevel Learning in Commognition

The commognitive approach is a conceptual and analytic framework that combines cognition and communication and views mathematics as a type of discourse. A discourse is defined as a type of communication characterized by its distinctive use of words and visuals, and what they endorse as narratives. A feature of discourse – routines – are defined as a task paired with a procedure, where a task is "any setting in which a person considers herself bound to act" based on her interpretation of the task and a procedure is the prescription for her actions in the task situation "that fits both the present performance and those on which it was modeled" from her past experience (Lavie et al., 2019, pp. 160–161). Routines are regulated by object-level rules (e.g., rules about addition regulates the routine of adding of fractions) or meta-level rules (e.g., rules that determine the use of "derivative" regulates the endorsement of statements about it).

The commognitive view of learning is changes in one's discourse, and teaching is considered as communicational activities with the motive of bringing students' discourse closer to what is considered canonic discourse (Sfard, 2008). Recent commognition studies considered teaching practice as "the task as seen by the performing teacher together with the procedure she executed to perform that task" (Nachlieli & Elbaum-Cohen, 2021, p. 7). They examined the teaching practices that promoted meta-level learning and called for more studies on this topic. Meta-level learning usually happens when students are engaged in activities regulated by meta-level rules they are unfamiliar with (e.g., proving or defining) (Park et al., 2023; Schüler-Meyer, 2020; Valenta & Enge, 2022) or when they encounter a new mathematical object to which old discursive rules do not apply (e.g., from whole numbers to fractions) (Cooper & Lavie, 2021). Teaching practices aiming to promote meta-level learning often use characteristics of students' old discourses as a starting point towards a new discourse to create discomfort with old discourse (Nachlieli & Elbaum-Cohen, 2021) or to allow students to use old routines in emerging discourse in a way appropriate in the teacher's eyes (Cooper & Lavie, 2021). We contribute to this area by examining teaching practices aiming to promote students' meta-level learning about new mathematical objects involving meta-level rules about multiple uses of a word "derivative".

Intellectual Needs

This study views teaching practice as teachers' actions to address students' intellectual needs, which emerge in a problematic situation that is "incompatible with, or presents a problem that is unsolvable by, his or her current knowledge" and the intellectual need is "the need to reach equilibrium by learning a new piece of knowledge" (Harel, 2013, p. 122). In commognitive terms, 'a problematic situation' can be seen as a task situation where students' existing routines do not work (i.e., do not lead to desirable results) or there is no existing routine they can apply. In this study, we considered task situations broadly in which students need to connect new words with their familiar words/symbols, realizations, narratives, and routines and to learn about how to communicate with the new words. Intellectual needs have been referenced in prior literature about teaching approaches to multiple uses of a mathematical word/symbol (e.g., Zazkis &

Kontorovich, 2016). Similar to those literature, we focused on three intellectual needs:

- Need for Causality: "one's desire to explain to determine a cause of the phenomenon"
- Need for Communication: the need to "externalize…an idea, or a concept, or the logical basis" for other to communicate with them
- Need for Structure: the need to organize "knowledge one has learned into a logical structure" (Harel, 2013, p. 126, p. 137, p. 140, respectively)

Since these needs are inextricably linked (Harel, 2013), we used them collectively to explain teaching practices aiming to address students' needs in their learning of multiple uses of a term.

Multiple Use of a Mathematical Word or Symbol and Teaching Approaches

As multiple uses of a signifier are prevalent in mathematics and cause potential difficulties for students, existing studies have investigated and suggested teaching approaches aiming to help students learn about such uses. Thompson and Rubenstein (2000) provided a list of such words including those that have more than one mathematical meaning (e.g., "square" in algebra and geometry). Rubenstein and Thompson (2001) provided a similar list of symbols with different meanings (e.g., "the raised -1 symbol" for "an inverse function" and a "reciprocal") (p. 268). Thompson & Rubenstein (2000) suggested teaching approaches to help students learn about such words/symbols for example by bringing their attention to different uses (e.g., a writing prompt, "I thought that a function was __. Now I know that a function is __.") and relations among multiple uses (e.g., "Square and cube have geometric meanings and are also used for second and third powers, respectively. How are the geometry and powers related?", p. 517).

Other studies that navigated teaching approaches specific to a word or symbol with multiple uses also focused on differences or relations among them. Lamb et al. (2012) pointed out difficulties that students have with multiple uses of the minus sign as "subtraction", "a symbol for a negative number" and "a unary operation of the opposite of" (p. 5) and suggested asking students to explain different uses of the minus sign in solution processes and providing numerical tasks that can expand to symbolic expressions (e.g., compare – –6 and –6, and then x and $-x$).

Biza et al. (2008) focused on multiple uses of "tangent line" that students encounter from Euclidian Geometry, Analytic Geometry, to Analysis and showed that many Analysis students (108 out of 196) still hold thinking developed in Geometry, "the tangent line is the one that has only one common point with the curve and leaves the curve in the same semi-plane" (p. 64) which do not generally apply to graphs of functions in Analysis. Biza et al. (2009) showed the similar results for teachers. The authors suggested providing students a strong image of "tangent line" from geometry and dominant but incorrect visual claims, and asking to verify algebraically.

Zazkis and Kontorovich (2016) analyzed secondary preservice teachers' (PSTs') responses to a hypothetical student questioning about the two uses of the exponent (-1) as the reciprocal and the inverse function. The PSTs addressed intellectual needs for structures for distinctive uses based on different contexts, terminologies, and locations of the symbol and related uses based on a common word ("inverse") that implies common actions (e.g., "undo") and also address the need for causality for why the same symbol is used in different contexts. They addressed intellectual need for communication using analogies to other symbols with multiple uses.

Two Uses of the Derivative

Our study deals with the two uses of the derivative, for each of the two mathematical objects, the derivative at a point and the derivative function, focusing on how they are different and related in the discourse about the derivative. For example, once the "the derivative at a point" is part of the discourse, it can be used to construct "the derivative function". There are many ways to do this, but one way is called *encapsulating* in Sfard's (2008) terminology. Specifically, once "the derivative at a point" is in the discourse, one can consider, for any number a, the pair $(a, f'(a))$. All the pairs can be collected into a set and encapsulated into a function called "the derivative function". Once the derivative function has been constructed, one can find its value at a specific input by substituting that value into the function. For example, the value of $f'(x)$ at $x = 1$ is $f'(1)$. We call this transition evaluating. Evaluating a function at a point is a process, typically done in calculus by substituting a value into a formula or reading a coordinate from a graph. In calculus, one of the important steps is appreciating the derivative at a point with the result of applying the evaluation process at a point to the derivative function as the same. Previous studies (Font et al., 2007; Park, 2013) have shown that the connection between evaluating the derivative function and the derivative at a point can be challenging for students.

Methods

Data Collection

The participating teacher, William (pseudonym) was educated in the U.S. He has B.S. and M.S. in Mathematics with over 10 years of teaching experience with 7 years of calculus teaching. We observed an Advanced Placement Calculus AB class with 34 students in a U.S. public high school. William used SMART board and generally organized his class starting with explaining key words and solving a few examples on the board, followed by students' individual or small group work on problems, and then whole class discussion about those problems. We video-recorded six 90-minute lessons at the beginning of the derivative unit, before the class began concentrating more on computation, where we could potentially observe the two uses of "derivative", how they were defined and connected to each other, and other realizations and real-life phenomena were discussed. We transcribed the videos including non-verbal communication.

Analysis

In our analysis, we treated the totality of the teacher's talk collected over multiple days as our unit of analysis (Sfard, 2008). However, we separated lessons into several episodes to see how words were used in different contexts. Episodes were defined by William's different teaching activities – e.g., defining a new mathematical term, providing a story involving a real-life object prior to defining a mathematical term, making connections between a newly defined term and previously defined terms, or making connections between a newly defined term and a real-life object – because changes in teaching activities correspond to changes in the context of discourse and, therefore, to potential changes in usage of words whose meaning is context-dependent (Biza & Zachariades, 2010; Thompson & Rubenstein, 2000), like "derivative".

Because we were interested in investigating potential teaching practices that aim to help students learn about multiple uses of the word "derivative" based on how William dealt with the feature, we first identified the mathematical objects that the common signifier signified in William's discourse. In class, William made an explicit distinction between two uses of the single word "derivative," namely as "a number" and as a "function", and we categorized William's use of the word "derivative" alone into two usages, namely as "the derivative at a point" or as "the derivative function," based on whether the surrounding context suggested it was being used as a number or as a function. We also recorded other terms and visual mediators (e.g., slope) used for those objects, and grouped episodes according to whether they included both objects or only one of them. We then recorded connections made between those terms and visual mediators. We operationalized teaching practice as a task–procedure pair (i.e., "the task as seen by the teacher together with the procedure she executed to perform that task") (Nachlieli & Elbaum-Cohen, 2021, p. 110872). Similar to other commognitive literature (e.g., Valenta & Enge, 2022), we first focused on William's actions related to the two uses of the derivative in the data, and wrote general instructional procedures based on these actions. We then identified the tasks William was trying to accomplish through these procedures based on intellectual needs.

Results

Teaching practices aiming to help students learn about the dual use of "derivative" based on our analysis of William's class are organized according to the intellectual needs they addressed.

Teaching Practice Addressing Intellectual Need for Structure and Causality

At the beginning of the derivative unit, William explicitly discussed multiple times how to distinguish the two uses of the signifier "derivative," as shown in the following excerpt observed on Day 2 after he constructed the derivative function graphically:

> This idea of the derivative can be thought of as a number, which represents the slope of the tangent line at a point. But, notice that I can talk about slope of the function at every point along this curve … And the slope is always changing. What we get then, is a function that represents the slope of the tangent line at any point as a function of x. That's called the derivative too. From the context, you understand what I'm talking about. …just remember now we're talking about numbers and we're talking about functions. We have to keep them straight. Then we get into the other half of Calc 1 and talk about integrals. There will be the same thing. There will be some integrals that are numbers, some integrals that are functions…we use the same words because the concepts are related.

Note that after using "derivative" to refer to both the derivative at a point and the derivative function, he provided a rule for distinguishing the two uses: if context shows the object signified by "derivative" is a number, then "derivative" signifies the derivative at a point and if context shows the object signified by "derivative" is a function, then "derivative" signifies the derivative function. This is a metarule because determining which object "derivative" signifies is something that participants in canonic discourse must do repeatedly even if often it is done tacitly. He also mentioned an equivalent metarule regarding how to distinguish the two uses of "integrals," another object with a similar feature that they were to cover soon. This addressed the students' intellectual need for structure for distinguishing the uses when the common word is used.

In the excerpt above, William also addressed intellectual need for connecting the two uses and causality of using one term for both. Specifically, using another object "slope" for both objects, he provided a connection between the two uses of the derivative – the derivative as a number and the derivative as a function – the latter comes into being when the former is expanded to several points over an interval. This connects the two uses via another mathematical object that are used for both objects and also shows how they are connected. He also provided the reason for using one word for two objects by stating "because the concepts are related."

Teaching Practice Addressing Intellectual Needs for Communication

Teaching practices addressing intellectual needs for communication were observed implicitly through the comparison among episodes. We identified the first teaching practice – connecting the dual uses of the derivative to a colloquial object with a similar feature – when William gave a story about a colloquial word with dual use by comparing an episode about "speed" and an episode about “derivative”. Specifically, he used a colloquial story about reading changing numbers on a speedometer to motivate the construction of the derivative function and followed a similar pattern as the colloquial story when constructing the derivative function. He used “the slope of the tangent line” at a point for both speed and the derivative and then considered them on an interval using numbers on a speedometer “changing all the time” similar to the slope “always changing” (see the excerpt in the previous section). Using the same structure and objects to discuss both speed and the derivative and talking about speed as a motivation to define the derivative seemed to address intellectual needs for communication because it allows students to use a new word “derivative” like they use a familiar word “speed” with the similar feature.

Another teaching practice addressed the need for communication. Specifically, the practice of avoiding the common term in one episode after the second object (the derivative function) is defined was observed with a shift in which object the common term “derivative” was used for, between the derivative at a point and the derivative function. In the beginning of the derivative unit, before the derivative function was defined, William used “derivative” dominantly for “the derivative at a point compared to other equivalent terms such as “slope of the tangent line,” or the symbol $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$. However, after defining “the derivative” as a function, he almost exclusively used the word “derivative” for the derivative function in episodes that involve both objects, except in the two episodes where the other object became included due to students’ questions or participation. Even in those cases, William responded to students using their wording or expressions, and went back to his previous use of the “derivative”.

It should be noted that once this shift in his use of “derivative” was made, he dominantly used “slope of the tangent line” to signify the result of evaluating the derivative function at a point. Specially, on Day 1, William said, “it [the derivative] is a function of any point I give you here, I can tell you what the slope of the tangent line is there”. Then, on Days 3 and 4 he solved five problems about this connection by computing “the slope of the tangent line” at a point given the equation for $f(x)$ by computing $f'(x)$ and evaluated it at a point. In this computation, only the expression $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ was used for the derivative function and only the term “the slope of the tangent line” and a symbol f' with a number (e.g., $f'(3)$) were used for the derivative at a point. In other words, in his class, the derivative function evaluated at a point became equivalent to the phrase “slope of tangent line” and the notation (e.g., “$f'(3)$”), but not to “the derivative at a point” in the way he initially defined it (e.g., “ $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$”).

Discussion and Conclusion

Our analysis of William’s class suggested several teaching practices aiming to help students learn about multiple uses of a mathematical term that addressed intellectual needs for:

A. Structure – Distinction: Discussing the metarules that distinguish the two uses of “derivative” – a number and a function
B. Structure – Connection: Using another mathematical object (slope) for both objects and showing how one use – the derivative as a function—can be engendered by the other use – the derivative as a number.
C. Causality: Using the connection between the two objects as a reason for using one term
D. Communication: Using the same structure to discuss a word for two mathematical objects and colloquial objects
E. Communication: Avoiding one word derivative in one episode once the second object (the derivative function) is defined

In general, those teaching practices are aligned with the teaching practices identified in the existing literature addressing multiple uses of a mathematical signifier, but also different from those due to the distinctive feature of the two uses of the derivative; the two objects for which "derivative" is used are usually used in the same context and one object is built up on the other object, which we elaborate on with teaching practices A and E only due to the limited space. Similar to the teaching practices addressed in other studies, teaching practice A provided a way to distinguish the two uses of the term "derivative" (e.g., the PSTs in Zazkis & Kontorovich, 2016 suggested what the raised (-1) is next to in order to distinguish between its use between "inverse" or "reciprocal"). However, the teaching practice we identified was making canonic discourse about the "derivative" explicit by stating when it is used as a number, it is "the derivative at a point," and when it is used as a function, it is "the derivative function". Some instructors may not explicitly discuss this metarule about how to distinguish multiple uses or may explicitly use the unabbreviated terms "the derivative at a point" and "the derivative of a function". However, given that students are learning about multiple uses of a common term when they are learning about a new concept, explicitly discussing the metarules of how to distinguish those two uses seems particularly important because one use is built up on the other and they are later related to each other. Being explicit about metarules about a new discourse (about the derivative, in our case) seems aligned with the teaching practices found in the literature for making the boundary between students' old discourse and new discourse explicit with students (e.g., rules about real numbers and rules about complex numbers, Nachlieli & Elbaum-Cohen, 2021). Communicating this could be further aided by providing students opportunity to directly attend to and reflect on the multiple uses of the derivative (e.g., writing prompts, "I thought that the derivative was __. Now I think that the derivative is ___," similar to Thompson & Rubenstein, 2000). The teaching practice E provides a way to communicate when multiple uses of the same term could in the same context. This was observed with a discursive shift in our data, where William initially used the "derivative" as the main signifier for "the derivative at a point", but once he defined the "derivative" as a function, he exclusively used the term for "the derivative function" whenever he talked about the two objects in one context. We note that such a shift could be seen as William's didactic choice, and there is an obvious appeal to avoiding the potential for confusion, ambiguity, and circularity inherent in using one term "derivative" for different objects at the same time. This appeal is magnified by the commognitive observation that student discourse is often initially a (potentially imperfect) imitation of instructor discourse (Sfard, 2014), which suggests that, even if the teacher's discourse carefully avoids such problems, students may not. Notably, given that this shift could be made implicitly by the instructor (as seen in our data), students may not even explicitly notice that the shift occurred in the teacher's discourse. This implicit nature of using "derivative" observed in the shift seems aligned with other literature reporting teachers' implicit shifts between different uses of the same word (Güçler, 2013 for "limit"). These discursive shifts would be interesting to further investigate because the literature has shown that the connection between the derivative at a point and the derivative function can be challenging for students (Font & Contreras, 2008; Park, 2013) and implicit shifts in instructor discourse have been tied to student difficulties (Güçler, 2013). Although derivatives have been intensely studied in the literature, we have not seen shifts like the one we discuss documented before. Note, however, that the instructors studied in (Park, 2015) also dominantly connected "derivative function" to "slope of the of the tangent line" and this leads us to conjecture that this shift may be widespread, but not documented because previous analyses were not looking for such large-scale discursive patterns.

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