Intro-to-Proof Students Discuss Logical Implication and Quantification: Themes from the Triangle Group

Alejandro Ortuno
Virginia Tech

Joseph Antonides
Virginia Tech

Benjamin Bruncati
Virginia Tech

Anderson Norton
Virginia Tech

Rachel Arnold
Virginia Tech

Keywords: proofs, logical implication, quantification, epistemological obstacle

Research has shown that students in introduction-to-proof courses encounter epistemological obstacles (EOs, or persistent challenges; cf. Brousseau, 2002) in learning and reasoning about core concepts, even in the face of research-based instruction (e.g., Norton et al., 2022, 2023). Quantification and logical implication are two major course topics that are seen for the first time in the case of most students; consequently, they tend to experience challenges when working with mathematical statements, which are then reflected as specific EOs. In our ongoing research project, we investigate students' experiences in introduction-to-proof courses with particular attention to eliciting and addressing students' EOs. Data from our project include whole-class discussions, one-on-one interviews with students, and small group discussions. In this poster, we present findings from our analyses of the discussions of one small group.

Our methods included analyzing video data from a group of students (the "Triangle Group") working on mathematical tasks during an undergraduate introduction-to-proof course. Tasks were drawn from existing research on the teaching and learning of mathematical proofs (e.g., Hub & Dawkins, 2018; Shipman, 2016). During the completion of these tasks, students in the small group were able to discuss their strategies and any mathematical insights or challenges that they experienced. Consistent with basic qualitative data-analytic methods (Merriam & Tisdell, 2015), we analyzed students' discussions and found patterns in students' reasoning—patterns that constituted the themes shared in this poster.

Three major themes emerged from our analyses. The first major theme involves students establishing constraints on the universal set when reasoning about mathematical statements. For example, the group was asked to negate the statement, "for all nonnegative real numbers x, $\sqrt{x} \leq x$." They responded, "for all nonnegative real numbers less than 1, $\sqrt{x} > x$," constraining the truth set to the interval (0, 1). The second theme involves students providing a counterexample when asked to find the negation of a statement. Using the same statement as above, the Triangle Group said the negation would be given if they "define x and write the exact same thing, [the statement] will be false." Specifically, they said P(x) is false if $x = 0.01$, finding a counterexample to the given statement. Lastly, the third theme explores how small group discussions can help students address EOs. For instance, students in the course were asked to consider how the statement, "the matrix A is invertible," is an existence statement. We inferred that this task elicited hidden quantification (Shipman, 2016), which is the EO caused when quantifiers are not explicitly stated in a mathematical statement. However, the Triangle Group seemed to be able to address this EO through their small-group discussion about the task.

Acknowledgments

This work was supported by the National Science Foundation under Grant No. 2141626. Additional support came from a 4-VA Collaborative Research Grant.

References

Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds.), *Theory of didactical situations in mathematics: Didactique des mathématiques*, *1970–1990* (pp. 79–117). Kluwer.

Hub, A., & Dawkins, P. C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *Journal of Mathematical Behavior*, *50*, 90-102.

Merriam, S. B., & Tisdell, E. J. (2016). *Qualitative research: A guide to design and implementation* (4th ed.). Jossey-Bass.

Norton, A., Arnold, R., Antonides, J., & Kokushkin, V. (2023). Epistemological obstacles related to treating logical implications as actions: The case of Mary. In T. Lamberg & D. Moss (Eds.), *Proceedings of the 45th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 71-80). University of Nevada, Reno.

Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2023). Addressing the cognitive gap in mathematical induction. *International Journal of Research in Undergraduate Mathematics Education*, *9*(2), 295-321.

Shipman, B. A. (2016). Subtleties of hidden quantifiers in implication. *Teaching Mathematics and its Applications*, 35(1), 41–49.

Intro-to-Proof Students Discuss Logical Implication and Quantification: Themes from the Triangle Group

Alejandro Ortuno alejandroo@vt.edu
Joseph Antonides jantonides@vt.edu
Benjamin Bruncati bbruncati23@vt.edu
Anderson Norton norton3@vt.edu
Rachel Arnold rachel.arnold@vt.edu

Motivation

- ***Epistemological obstacle (EO):*** a persistent challenge in one's learning, even in the fact of best instruction (Brousseau, 2002; Norton et al., 2023)
- Research shows students encounter specific EOs related to core concepts about proof and proving (Arnold et al., in press; Norton et al., 2022, 2023; Shipman, 2016)
- The Proofs Project aims to identify EOs and design instructional strategies for supporting students by eliciting and addressing EOs directly

Methods

- The Proofs Project collected data from whole-class discussions, small-group discussions, and clinical interviews with students
- Tasks designed by The Proofs Project or adapted from published research (e.g., Dawkins & Roh, 2020; Hub & Dawkins, 2018; Vroom, 2020)
- Regular opportunities for discussion provided in class for students to share ideas, strategies, and challenges
- Analyzed video data from small groups of students working on tasks during an intro-to-proofs class
- Focus of this presentation on the "Triangle Group"
- Identified regularities in qualitative findings from our analysis of small-group discussions, constituting the themes shared here (Merriam & Tisdell, 2016)

Theme 1

Students place constraints on the universal set when reasoning about mathematical statements.

Task 1

Determine if the following statement is true or false. "Suppose $a, b,$ and c are integers. If $|a-b|<5$ and $|b-c|<5$, then $|a-c|<5$."

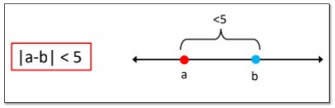

Fig. 1: Visual representation for Task 1

"Since it's an absolute value, we know at least one of them [a or b] has to be <u>over 5</u>. Or... I don't know. There's some sort of <u>constraint</u> about a and b, like <u>over 5 or under 5</u>, or something like that."

Task 2

Negate the statement. "For all nonnegative real numbers x, $\sqrt{x} \le x$."

"For all nonnegative real numbers <u>less than 1</u>, $P(x)$ is false."

Constrained the universal set to the interval (0,1).

Theme 2

Students provide a counterexamples when asked to find the negation of a statement.

On Task 2, Triangle Group initially found a counterexample to the statement.

"Don't we just need to find a value of x where the square root of that number is greater than? ... $P(.01)$ is false"

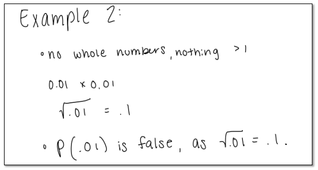

Fig. 2: Group's work on Task 2

Progressing from identifying a counter-example to writing a negation of the statement posed a challenge.

- **S1: So then the whole statement is false. Therefore, $P(x)$ is false.**
- **S2: So, in order to make this statement true—**
- **S3: Well, $P(x)$ is the open statement. So, all of $P(x)$ isn't false. 'Cause, $P(x)$ is true for like 2 or 3. So I think we can just say that, um, just statement P of .01 is false.**

Theme 3

Small group discussions can help students elicit EOs and work together to begin addressing EOs.

Task 3

Explain how the statement "Matrix A is invertible" *is an existence statement.*

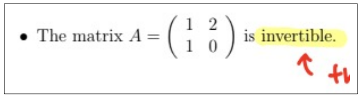

Fig. 3: Statement for Task 3

Task designed to elicit the EO of hidden quantification (Shipman, 2016).

"It's kind of like saying, *there exists* a matrix such that [A] is invertible."

Group member unveils the hidden quantifier, reframing the statement as an existence statement.

Task 4

Prove or disprove the statement. "If x > 0, then $x + \frac{1}{x} \ge 2$."

Statement contains a hidden universal quantification. Through working together, the group was able to unveil the universal quantifier and work together to negate the statement.

"...the way you would negate it would be 'there exists', right?"

Purpose

To share themes from qualitative analysis of one small group's discussions about research-based tasks during class

References

Arnold, R., Kokushkin, V., Norton, A., & Antonides, J. (in press). Students' treatment of the negation of logical implication. To appear in Proceedings of the 26 annual conference on Research in Undergraduate Mathematics Education.

Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds.), *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990* (pp. 79–117). Kluwer.

Dawkins, P.C., & Roh, K.H. (2020). Assessing the influence of syntax, semantics, and pragmatics in student interpretation of multiply quantified statements in mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *6*(1), 1-22.

Hub, A., & Dawkins, P.C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *Journal of Mathematical Behavior*, *50*, 90-102.

Merriam, S.B., & Tisdell, E.J. (2016). *Qualitative research: A guide to design and implementation* (4th ed.). Jossey-Bass.

Norton, A., Arnold, R., Antonides, J., & Kokushkin, V. (2023). Epistemological obstacles related to treating logical implications as actions: The case of Mary. In T. Lamberg & D. Moss (Eds.), *Proceedings of the 45th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 71-80). University of Nevada, Reno.

Norton, A., Arnold, R., Kokushkin, V., & Tiraphatna, M. (2022). Addressing the cognitive gap in mathematical induction. *International Journal of Research in Undergraduate Mathematics Education*, *9*(2), 295-321.

Shipman, B.A. (2016). Subtleties of hidden quantifiers in implication. *Teaching Mathematics and its Applications*, 35(1), 41–49.

Vroom, K. (2020). Guided reinvention as a context for investigating students' thinking about mathematical language and for supporting students in gaining fluency [Unpublished doctoral dissertation]. Portland State University.

Acknowledgements

Support for this work was provided by NSF Grant DUE-2141626 and by a 4-VA Collaborative Research Grant.

The Proofs Project Website