Graduate Students' Pedagogical Mathematical Practices Used in Approximations of Teaching Practice Tasks

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This research design uses Approximations of Practice (AoPs), simulated practices of responsive pedagogy for prospective teachers to respond to, which lead them to make relevant connections between advanced mathematics courses and teaching practices. Data were collected from students in a Master's level Mathematics for Teachers course. The curriculum and interviews focused on three mathematical content domains: probability, algebra, and analysis. The AoPs were written into a semi-structured interview to demonstrate real-world examples of how teachers may see this material arise in the classroom. The AoPs prompted the participants to interpret and respond to student thinking and structure whole class discussions about students' strategies. Their responses were analyzed using Wasserman's (2022) pedagogical mathematical practices (PMPs). We examined each response to the AoPs to identify the participants' PMPs.

Keywords: Pedagogical Mathematical Practices, Advanced Math, Approximations of Practice

Learning advanced mathematics is an essential part of teachers' preparation, as it can help them understand relations among mathematical concepts and refine their mathematical practices, which can inform various pedagogical practices (e.g., Baldinger, 2018; Serbin, 2021; Zazkis & Kontorovich, 2016; Zazkis & Marmur, 2018; Zbiek & Heid, 2018). The Conference Board of the Mathematical Sciences (2012) recommended that there be opportunities in mathematics courses for prospective teachers to make explicit connections between advanced mathematics and the teaching of secondary mathematics. However, these connections between the curricula and teaching are often left unstated (Wasserman & Weber, 2017), so it is essential to make them more explicit. Researchers and educators have made such explicit connections between advanced and secondary content in their innovative curricula (e.g., Álvarez et al., 2020; Burroughs et al., 2023; Fukawa-Connelly et al., 2020; Goar & Lai, 2022; Wasserman et al., 2017).

Following these efforts, we designed a Master's-level Mathematics for Teachers course on connections from abstract algebra, real analysis, and probability to the teaching and learning of secondary / college mathematics. In this course, we implemented Approximations of Practice (AoPs; Grossman et al., 2009), in which the students, who were preservice or in-service teachers, simulated teaching practices of noticing students' mathematical thinking (Jacobs et al., 2010) and orchestrating class discussions about students' ideas (Smith & Stein, 2018). AoPs are productive in supporting prospective teachers in using their *knowledge* of advanced mathematics in their teaching practices (e.g., Álvarez et al., 2020; Burroughs et al., 2023; Serbin, 2021). AoPs can also be productive for supporting prospective teachers in connecting the *mathematical practices* developed in their mathematics courses to their teaching. Wasserman (2022) intersected mathematical practices (Rasmussen et al., 2005) with pedagogical practices (National Council of Teachers of Mathematics, NCTM, 2014) to develop the theoretical construct of *Pedagogical Mathematical Practices* (PMPs), which are the disciplinary practices common to mathematicians and teachers. There is a need for researchers to identify PMPs that can be enacted in the classroom and AoPs. We address this need with this research question: *What PMPs do graduate prospective and in-service teachers engage in as they work on AoPs?*

Literature Review

AoPs are a valuable tool for engaging prospective teachers in pedagogical situations before entering the classroom. *AoPs* are "opportunities to engage in practices that are more or less proximal to the practices of a profession" (Grossman et al., 2009, p. 2056). AoPs can be enacted by teachers acting out classroom scenarios, interacting with simulated students in mixed reality simulations, interpreting written student work, or writing scripts for a class scenario. AoPs tend to require teachers to simulate a response to student thinking using pedagogical practices, such as noticing students' mathematical thinking (Jacobs et al., 2010), responding to hypothetical class situations (Zazkis & Marmur, 2018), and scripting discussions (e.g., Zazkis et al., 2013).

Instructors can use AoPs in their mathematics content courses and teaching methods courses for future teachers. For instance, Tyminski et al. (2014) used AoPs to support prospective elementary teachers' practice of organizing discussions around the different students' strategies. Campbell and Elliott (2015) created an AoP simulating a class discussion to define trigonometric ratios in a high school geometry class by introducing a problematic mathematical situation. These studies illustrate how engaging prospective teachers in AoPs can give them experience using certain pedagogical practices before they teach.

AoPs can bridge teachers' learning of advanced mathematics to their teaching (e.g., Álvarez et al., 2022; Burroughs et al., 2023; Lischka et al., 2021). Álvarez et al. (2020) suggested that teaching is a form of applied mathematics, so teaching applications should be incorporated into undergraduate mathematics courses' curricula, just as engineering and physics applications often are. Other researchers have used scripting tasks as simulations of practice to help students develop productive mathematical understandings (e.g., Marmur & Zazkis, 2022; Zazkis & Cook, 2018) and make connections between advanced mathematics and the teaching of secondary mathematics (e.g., Fukawa-Connelly et al., 2020; Serbin & Bae, 2023; Wasserman et al., 2017). Other researchers have used noticing tasks (Jacobs et al., 2010) to connect advanced and secondary mathematics. For example, Serbin (2021) demonstrated how prospective teachers used their abstract algebra knowledge as they interpreted and decided how to respond to hypothetical students' mathematical thinking in noticing tasks. Overall, researchers have documented the value of AoPs when embedded in the study of advanced mathematical topics. In this study, we further contribute to the research based on AoPs by analyzing the PMPs of graduate students, who are prospective or in-service mathematics teachers, as they engage in AoPs.

Theoretical Background

"Teaching relies on more than just knowledge" (Wasserman, 2022, p. 29); it also relies on practices. Just as researchers have explicated the intersection of mathematical knowledge and pedagogical knowledge (e.g., Shulman, 1987), Wasserman (2022) conceptualized the construct of Pedagogical Mathematical Practices (PMPs) as an intersection of mathematics and pedagogy with respect to practice. Practices of the discipline are the "regular actions, activities, habits, behaviors, processes, norms, etc. that one engages in while 'doing' that activity" (Wasserman, 2022, p. 5). Wasserman explored this domain of practice through the intersectional lens of mathematical practices (MPs) and pedagogical practices (PPs). MPs are the practices or mathematical habits of mind (Cuoco et al., 1996) that mathematicians engage in, such as identifying patterns, proving, conjecturing, problem-solving, and defining (e.g., Heid et al., 2015; Polya, 1945; Rasmussen et al., 2005). PPs are the practices teachers engage in as they teach mathematics, such as those outlined by NCTM (2014) and Hunter et al. (2016). The intersection of MPs and PPs are *PMPs:* "the regular actions, activities, habits, behaviors, processes, norms, etc., that are productive both as a mathematical practice and as a pedagogical practice for teaching mathematics" (Wasserman, 2022, p. 30). Examples include:

> Acknowledge and revisit assumptions and mathematical limitations; consider and use boundary cases to test and illustrate mathematical ideas; expose logic as underpinning mathematical interpretation; use simpler objects to study more complex objects; avoid giving rules without accompanying mathematical explanations; and seek out and use multiple explanations. (p. 30)

In addition to the utility of this construct for analyzing mathematics teachers' practices, the PMP construct also has useful implications for mathematics teacher preparation. PMPs have the potential for integrating pedagogy into advanced mathematics courses and for integrating advanced mathematics into pedagogy methods courses. This can contribute to preservice and in-service mathematics teachers perceiving more connections between advanced mathematics and the teaching of school mathematics. Given the need for researchers and mathematics teacher educators to support teachers in making such meaningful connections between advanced mathematics and the teaching of school mathematics, there is a need for researchers to identify additional PMPs and explore the contexts in which teachers use them. Our study contributes to this need through our identification of PMPs used by teachers as they engage in AoPs.

Methods

We recruited six participants from a large public research university in the southern US, Jessica, Roberto, Amy, Selena, Linda, and Eduardo (pseudonyms), who were enrolled in an MS in Mathematics program with a Math Education concentration. Jessica and Roberto were in-service teachers with multiple years of mathematics teaching experience. The others were preservice teachers who intended to teach high school or college. All participants had recently completed a Master's-level Mathematics for Teaching course, which addressed content from abstract algebra, real analysis, and probability with connections to the teaching of secondary/ college mathematics. The course instructors often assigned AoPs in class work and assignments.

The second and third authors conducted semi-structured task-based clinical interviews (Drever, 1995; Clement, 2000) with each participant two weeks after they completed their course. The interviews were conducted over Zoom and were recorded and transcribed for retrospective analysis. The participants performed the AoPs shown in Figure 1. These tasks were designed to simulate the practices of noticing/ attending to, interpreting, and deciding how to respond to students' mathematical thinking and leading class discussions about students' work. The four AoPs were each related to the mathematical content that the participants had learned during the Mathematics for Teachers course: conditional probability, the differentiability of functions, the zero-product property, and inverse functions. We designed the interview tasks to elicit evidence of the participants' PMPs used in their responses. We asked the participants questions related to what they noticed in the students' responses, what they interpreted about the students' understandings, how they would respond to the students as their teacher, and how they would facilitate a class discussion about the students' approaches. The first and second authors performed inductive coding on the interview transcripts (Miles et al., 2014), in which we open-coded PMPs that were evident in the responses to the AoP tasks. A participant's response was indicative of a PMP when it satisfied Wasserman's (2022) criteria: the practice was common to mathematicians and mathematics teachers. We used Wasserman's (2022; 2023) PMPs as a list of potential a priori codes to begin the creation of our codebook, and we added newly found PMPs to the codebook. We recoded each response to the AoP tasks in two rounds of deductive coding (Miles et al., 2014). We identified patterns among the coded PMPs across the responses, which provided insight into their similar or varied usage of PMPs.

a. AoP Task 1

In an introductory probability class, your students are discussing the following problem on a coin toss game.

> In this game, two players toss a fair coin 4 times each and compare who gets the most heads to determine the winner. Player A tossed the coin 4 times and got 2 heads. Suppose Player B tossed the coin 2 times and got 2 heads. Player B will toss the coin 2 more times and needs one more head to win the game. What do you think about the probability that Player B will win the game?

Read the following discussion between three students.

Student 1: Well, Player B already got two heads in a row. So, I guess it is much less likely to get a head in the remaining two attempts, because, you know, you would expect two heads on average out of four tossing in anyways, right?

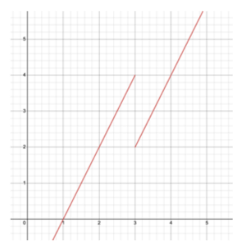
Student 2: We should think about all possible outcomes for tossing the coin four times. Each time, we have two possible outcomes of heads or tails, and repeat it four times, so there are $2 \times 2 \times 2 \times 2 = 16$ possible outcomes in total. These 16 outcomes are all equally likely, so the probability to get each is exactly $1/16$. Player B already got two heads and needs at least one more heads, so the only possible cases are $HHHT$, $HHTH$, $HHHH$. Since we have only 3 cases to win and each has the probability of $1/16$, the probability of winning the game is $3/16$.

Student 3: We know Player B will need at least one head in the remaining two attempts. Since the coin is fair, there will be 25% chance of getting two tails (TT) and 75% chance of getting one or two heads (HT, TH, HH). So, the chance of winning for Player B at this point is 75%.

b. AoP Task 2

In the discussion, three students explained whether the function is differentiable or not and why they think so.

$$f(x) = \begin{cases} 2x - 4 & (x \geq 3) \\ 2x - 2 & (x < 3) \end{cases}$$



Student A: This function is differentiable because I can see the two lines of its graph are parallel, so this function has the same slope everywhere including the point $x = 3$.

Student B: This function is differentiable because I can take the derivative of $2x - 4$, which will be 2 and then it's the same for the derivative of $2x - 2$. You just take the coefficient of the x. So, the derivative of this function is 2.

Student C: This function is not differentiable because its graph is not connected. We know continuous functions are differentiable.

c. AoP Task 3

Suppose you are teaching an Algebra II class on solving quadratic equations. While students work on the task of solving $x^2 + x - 6 = -4$, you see different students use these three approaches.

Student 1	Student 2	Student 3
$x^2 + x - 6 = -4$ $(x-2)(x+3) = -4$ $x - 2 = -4, \quad x + 3 = -4$ $x = -2, \quad x = -7$	$x^2 + x - 6 = -4$ $x^2 + x - 6 + 4 = -4 + 4$ $x^2 + x - 2 = 0$ $(x+2)(x-1) = 0$ $x + 2 = 0, \quad x - 1 = 0$ $x = -2, \quad x = 1$	$x^2 + x - 6 = -4$ $(x-2)(x+3) = -4$ $x - 2 = 2, \quad x + 3 = -2$ $x = 4, \quad x = -5$

d. AoP Task 4

Suppose you are teaching a Calculus I class on derivatives of trigonometric and inverse trigonometric functions. Your students are working on the task of finding the critical points of $f(x) = \cos^{-1}(x)$. You see different students use these two approaches for finding the critical points of $f(x) = \cos^{-1}(x)$.

Student 1	Student 2
$f(x) = \cos^{-1}(x)$ Use the power rule and chain rule: $f'(x) = -1 \cdot \cos^{-1-1}(x) \cdot -sin(x)$ $f'(x) = \cos^{-2}(x) \cdot sin(x)$ $f'(x) = \frac{sin(x)}{\cos^2(x)}$ Set this equal to 0. $\frac{sin(x)}{\cos^2(x)} = 0$ $sin(x) = 0$ Critical points: $x = 0 + k\pi$ for $k \in \mathbb{Z}$	$f(x) = \cos^{-1}(x) = \frac{1}{\cos(x)} = \sec(x)$ $f'(x) = \frac{d}{dx}\sec(x)$ $f'(x) = \sec(x)\tan(x)$ Set this equal to 0. $\sec(x)\tan(x) = 0$ $\sec(x) = 0$ or $\tan(x) = 0$ $\sec(x) = 0$ has no solution. $\tan(x) = \sin(x)/\cos(x) = 0$ Critical points are anything that makes $\sin(x) = 0$ but $\cos(x) \neq 0$ Critical points: $x = 0 + k\pi$ for $k \in \mathbb{Z}$

Figure 1. AoP Tasks Used in Interviews

Results

Overview and Examples of the Preservice and In-Service Teachers' PMPs

In the four AoPs, 15 PMPs were elicited (see Figure 2). PMP 1 was modified from Wasserman's (2022) PMP, "Acknowledge and revisit assumptions and mathematical limitations" (p. 8), and PMP 9 came directly from Wasserman's (2022) findings. We identified 13 new PMPs using inductive coding (see Figure 2). PMPs 1, 2, 3, and 7 were used the most frequently by all participants. PMPs 1, 2, and 3 rely on participants noticing and attending to student thinking. After noticing student thinking, participants frequently validated the mathematical procedure by identifying a mathematical concept that made the work correct or incorrect (PMP 7). An example of using these four PMPs in one AoP can be seen in Roberto's response to AoP 2 (see Figure 1b). The interviewer asked what Roberto noticed about the students' reasoning, and he said:

> Student A says that it has the same slope. I don't know if he's talking about the tangent, but that has nothing to do with this case. It's not really helping to answer the question if it's differentiable. The second student is just taking the derivatives of the piecewise function for each part and getting two, so he's just assuming that this is the same for the derivative, then it's going to be differentiable… differentiable functions are continuous, but the converse of that is not always true.

Roberto acknowledged assumptions (PMP 1) made by the students regarding slope and then invalidated the logic of that claim (PMP 2) by stating it did not answer the question about differentiability. The participant interpreted what the second student was doing in their work (PMP 3) and finally identified the concept of continuous and differentiable functions to invalidate Student C's assumption about differentiability (PMP 7).

Linda used PMP 11 regarding simplifying mathematical definitions into their own words when responding to AoP 2 (see Figure 1b). The interviewer asked if she would ask the students any specific questions to relate it back to the definition. She said,

> I would ask the class to maybe put the definition in their own words because I think sometimes definitions are a little bit fancy. To kind of reason them putting them into their own words would maybe help them connect what they're trying to say to the definition.

Simplifying a mathematical definition in your own words is a valuable tool mathematicians use to better understand a definition. Linda leveraged that PMP in her pedagogical techniques.

An example of PMP 15 was used in Roberto's response to AoP 3 (see Figure 1c). When asked how he would have the students recognize which one was wrong, he responded, "Probably put them in groups and work it out and see what the majority of the group gets or the class gets. If they get it right, I will use that definitely as the answer." PMP 15 was used seven times, mostly in response to the probability group discussion in AoP 1 (see Figure 1a).

Jessica shows an example of using PMP 12 in response to AoP 3, which asked her what she could tell about the students' mathematical understanding (see Figure 1c). Jessica said:

> Student 1 doesn't understand the clear definition of the product. If two things multiplied together, they can't each be -4, because that would be positive 16 there. While Student 3 understands the definition of a product, I feel like they don't understand while $2 \times (-2)$ does work, there's not a value of x that's the same that will work to give you that -4.

Jessica's response identified the procedural error in the student's work (PMP 12), which is a practice that both mathematicians and teachers would do when looking at their own or others' work. Jessica also interpreted the student's thinking (PMP 3), identified errors in their mathematical logic (PMP 2), recognized the concepts that validate a student's use of multiplication and factorization (PMP 7), and checked the student's work (PMP 6). This example shows how multiple PMPs can be used in a short response.

1. Acknowledge assumptions and/or mathematical limitations.
2. (In)validating the logic of a mathematical claim.
3. Interpreting details in someone's mathematical claim.
4. Referencing definition/formulas to verify a mathematical argument.
5. Using multiple representations to make sense of a concept.
6. Check work.
7. Identifying the mathematical concept that validates a mathematical procedure.
8. When multiple mathematical procedures are available choose the most efficient and accurate procedure.
9. Use simpler objects to study more complex objects.
10. Finding a counterexample to a mathematical claim.
11. Simplifying a mathematical definition in your own words to demonstrate understanding.
12. Identifying procedural errors in mathematical work.
13. Interpreting and understanding mathematical symbols and language.
14. Comparing and contrasting mathematical objects or concepts.
15. Combining multiple people's explanations to find a valid mathematical approach.

Figure 2. List of PMPs identified in the data.

Table 1. PMPs Used by Each Participant in Each AoP

	Probability AoP	Differentiability AoP	Equation AoP	Inverse AoP
Amy	1, 2, 3, 4, 5, 6, 7, 11, 12, 14, 15	1, 2, 3, 4, 5, 6, 7, 10, 11, 14	1, 2, 3, 4, 6, 7, 8, 9, 10, 12	1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14
Eduardo	1, 2, 3, 7, 14, 15	1, 2, 3, 4, 7, 10, 14	1, 2, 3, 5, 6, 7, 10, 11, 12, 14	1, 2, 3, 5, 7, 9, 12, 13, 14
Jessica	1, 2, 3, 5, 6, 7, 11, 14	1, 2, 3, 4, 5, 10	1, 2, 3, 6, 7, 8, 12, 14	1, 2, 3, 5, 6, 7, 12, 13, 14
Linda	1, 2, 3, 7, 10, 15	1, 2, 3, 4, 7, 11	1, 2, 3, 6, 7, 8, 9, 12, 14	1, 2, 3, 5, 6, 7, 9, 14, 15
Roberto	None	1, 2, 3, 4, 7	1, 2, 3, 6, 7, 12, 14	1, 2, 3, 5, 7, 9, 10, 13, 14
Selena	1, 2, 3, 4, 7, 14, 15	1, 2, 3, 4, 5, 7, 10, 11, 14	1, 3, 6, 7, 12, 14, 15	1, 2, 3, 4, 5, 7, 13, 14

All participants used at least 13 of the 15 identified PMPs throughout the four AoPs, but each participant had some nuances in their use of PMPs. The PMPs used by each participant in each AoP are presented in Table 1. Amy consistently drew on multiple PMPs throughout the whole interview. She used 84 PMPs to answer 11 interview questions. In comparison, Roberto only used 33 PMPs. The lower number of PMPs used could be partly attributed to his uncertainty in his ability to solve the probability problem, which led him to not respond to the probability related AoP. Roberto focused heavily on the PMPs 1, 2, 3, and 7 that involved attending to student thinking and validating the mathematical procedures with the correct mathematical concept. Although he could use other various PMPs, he only leveraged a small subset of PMPs in the interview setting. There are common PMPs, such as 1, 3, and 7, that were used by each participant to respond to all class discussion tasks. Some PMPs were more prevalent in certain AoPs; for example, on the Probability AoP, the participants commonly used PMP 14 and PMP 15. They compared the students' approaches and combined their responses to lead a productive classroom discussion. For the analysis task, PMP 4 was elicited by 5 of the participants. Participants found it helpful to use the definition to answer the discussion task. That analysis-related AoP 2 also showed the highest use of PMP 10, where the participants used a counterexample to prove the student response was incorrect. We can see the highest use of PMP 6 for the algebra-related AoP 3 where 4 teachers used it to respond to the discussion task. This task asked students to find the zeros of a quadratic equation, and checking work was the most used PMP by the participants. PMPs 13 and 14 were the most frequently used for the algebra and analysis calculus-based task, in which students misinterpreted the meaning of the -1 superscript. PMP 13 involved interpreting and understanding mathematical symbols, and PMP 14 involved comparing the students' work as a tool to have a classroom discussion. Thus, certain PMPs were used more frequently depending on the mathematical content addressed in the task.

Participants' Use of Certain PMPs Seemed to Depend on the Type of Pedagogical Task

Each subsequent AoP question was intended to prompt the participants to use the information they identified in a previous task to inform their decision-making for the subsequent class discussion AoP. In the initial questions, where the interviewer asked the participant to identify what the students were doing in their work, the participants relied heavily on PMPs 1, 2, 3, and 7. These PMPs require noticing and interpreting student work. As the tasks progressed to the questions regarding how they would respond to the students or lead a classroom discussion about the students' ideas, the participants drew on their decisions from the prior tasks and utilized more PMPs. This is evident in Amy's response to AoP 1. Her response to the question about what she noticed about students' thinking used PMPs 1, 2, 3, 7, and 14. Her response to a question about how she would respond to the students involved her additional use of PMPs 4 and 10. In her response to the question about how she would facilitate a class discussion about the students' ideas, Amy used PMPs 1, 2, 3, 4, 5, 6, 7, 11, 12, 14, and 15. The subsequent addition of different PMPs used as the questions in each AoP task progressed gives evidence to the effectiveness of the task design in eliciting teachers' use of PMPs. Wasserman (2022) claimed, PMPs "can be used to structure discussions of, and give insight into, observed episodes of teaching and practicum experiences" (p. 13). Our research suggests that PMPs are a tool to help teachers structure their classroom discussions and attend to student thinking. As the tasks progressed, participants made decisions on the information presented in the earlier part of the AoP and used PMPs to structure a productive classroom discussion. The participants frequently used five PMPs for these discussion tasks, PMPs 1, 2, 3, 7, and 14. These PMPs may be particularly conducive for teachers leading classroom discussions about students' mathematical thinking.

Discussion and Conclusion

This study showed how the participating teachers used PMPs in AoP tasks where they engaged in hypothetical classroom situations. The use of PMPs implies that their experience in the course encouraged them to connect their advanced MPs with AoPs in the classroom. Prior to this study, the teachers learned advanced topics in real analysis, abstract algebra, and probability by engaging in MPs like defining concepts in abstract algebra and examining special examples of functions to understand the relationship between continuous functions and differentiable functions in real analysis. They applied those practices in the AoP tasks in this course. Their use of PMPs in the interviews conducted two weeks after the semester indicates that the design of the course contributed to preparing the teachers to carry over their practices from advanced mathematics into their future classroom teaching, which could be further examined by following their classroom teaching as shown in prior studies (e.g., Wasserman & McGuffey, 2021). These findings support the participating teachers' perceived changes in their views on the relevance of learning advanced mathematics to mathematics teaching by providing evidence of their demonstration of PMPs in the AoPs. This study gives evidence of their applications of practices in advanced mathematics to hypothetical classroom scenarios. Future studies are needed to examine how teachers use PMPs in actual classroom settings and to further investigate different patterns of using PMPs by content areas and by individual teachers.

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