Student Understanding of the Direction of Vector Dot Products Across Contexts and Levels

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As part of an effort to examine students' understanding of vector products, we present preliminary findings from student responses to survey questions asking whether the dot product has a direction in three different contexts and three different class levels. The contexts were two vectors with labels but no values, two vectors on a Cartesian grid with specified magnitudes and directions, and the flux of a uniform electric or magnetic field through a planar surface. The three courses were the second semester of an introductory calculus-based physics course, a sophomore-level mathematical methods in physics course, and a junior-level electricity and magnetism course. Depending on the context, between 25% and 30% of introductory students were successful at determining that the dot product has no direction. Performance increased with course level.

Keywords: Vector Products, Student Thinking, Pseudo-longitudinal

Introduction

Vectors are central to building a robust understanding of physics. Both conceptually and computationally, vectors play important roles through all levels of physics instruction. This has resulted in a significant body of literature addressing student understanding of vectors (e.g., Flores et al. (2004), Gire and Price (2014), and Nguyen and Meltzer (2003)). As is the case with the majority of physics education research, the majority of these studies are conducted in introductory calculus-based physics courses (Kanim & Cid, 2020). While some work has examined student understanding of vector fields (Bollen et al., 2017; Küchemann et al., 2021) and differential vector products (Topdemir et al., 2023; Walker & Dray, 2023) in the upper division, the evolution of students' ideas about vector products (dot and cross products) has not been thoroughly explored. This paper examines a preliminary study of the evolution of students' ability to identify properties of vector products at different levels and in different contexts. Data from a pseudo-longitudinal study is presented to demonstrate some aspects of how student performance is related to these factors.

Background

A significant amount of the work on student thinking about vectors has focused on student difficulties, with computational difficulties taking center stage. Mikula and Heckler (2013) examined students' ability to breakdown vectors into individual components by utilizing appropriate trigonometric operations. The decomposition of vectors into components has been shown to be challenging for students from a variety of introductory physics courses (algebra- and calculus-based, first and second semesters). Heckler and Scaife (2015) explored the effects of representation on student performance when adding and subtracting vectors, replicating previous findings on student difficulties adding vectors in the graphical (arrow) representation and identifying some novel difficulties. They also found that students performed significantly better using an algebraic representation than a graphical one. Bollen and colleagues (2017) examined student difficulties around the interpretation, construction, and translation between different representations of vector fields. They found that students' difficulties with addition and decomposition of vectors also impacted students' success transitioning between symbolic and graphical representations of vector fields and noted difficulties utilizing field line representations to appropriately represent field strength. A general lack of representational fluency was identified as a reason students may struggle with determining properties of fields such as the signs of the divergence and/or curl.

Zavala and Barniol (2013) investigated student understanding of the dot product in a graphical representation in introductory physics courses. This was done in three distinct contexts: mathematical (labeled "no context"), mechanical work, and electric flux. No significant difference was found between the two physical contexts, however there was lower performance on the no context problems than the others. Students also had significant difficulties connecting a conceptual understanding of the dot product to its formal representation. Barniol and Zavala (2014) subsequently produced the 20-item, multiple-choice Test of Understanding of Vectors (TUV), by considering difficulties reported by a large pool of previous studies and generating incorrect answers that tied directly to this literature. The TUV was found to track onto four primary factors for student difficulties: graphical properties, graphical procedures, geometric calculations, and unit vector notation calculations. However, the TUV did not include any physical contexts, and although the TUV did include some questions on the meaning of vector products, the majority of the items focused on procedures.

More recently, Carli and colleagues (2020) developed the Test of Calculus and Vectors in Mathematics and Physics (TCV-MP), examining differences in students' ability to answer questions in mathematical and physical contexts by asking 17 pairs of isomorphic questions. They found that students did not always use the same strategies to solve the isomorphic questions, implying that context influenced students' framing of the questions. Different difficulties were demonstrated by students between some question pairs. The researchers argue that differences in performance on some of the items suggest that success in physics is not solely dependent on mathematics preparation, but also necessitates students be able to appropriately blend the mathematical and physical reasoning in their problem solving. This is supported by other literature on student reasoning (e.g., Kuo et al. (2013) and Uhden et al. (2012)).

As part of a larger project to investigate student understanding of vector products at different levels and in different contexts, here we focus on recognition of features of the dot product. The research question we are pursuing here is: How, and in what ways, does student understanding of the direction of vector dot products differ between mathematics and physics contexts, and at different levels in the physics curriculum?

Methods

Data were collected via a four-item, online, multiple-choice survey, with three items focusing on dot products. While the result of the dot products are scalars and do not have a direction, it is not uncommon for physics students to ascribe directions to dot products and quantities described by dot products. Questions in the literature that ask students to calculate the dot product had both vector and scalars as part of the multiple-choice options (Carli et al., 2020). With common incorrect responses being vector options, it is unclear how students understand the nature of the operation of a dot product. Additionally, the selection of a calculated scalar response does not explicitly imply an understanding that the dot product has no direction. To address the vector versus scalar nature of vector products we asked about the magnitude and direction of the products in separate questions. Other questions from the literature include the words "vector" or "magnitude" in the answer (Barniol & Zavala, 2014). To avoid any cueing from specific vector-related vocabulary we asked, "Does the dot product $\vec{A} \cdot \vec{B}$ have a direction, and if so, what is it?"

Since the dot product results in a scalar quantity, the correct answer is, "It does not have a direction." The answer choices were based on distractors from prior literature (Barniol & Zavala, 2014; Van Deventer, 2008). In all items students were presented with the appropriate formula for the dot product and asked whether it has a direction. A generalized list of the answer choices and abbreviated labels for presentation purposes can be found in Table 1. The same phrasing was used in all direction questions; only the vector labels were changed to reflect the specific context.

Table 1. Generalized multiple choice responses and abbreviated codes for the survey items.

Answer choices	Abbreviation
It does not have a direction	ND
In the direction of $\vec{A}$ (field vector $\vec{E}$ or $\vec{B}$ in the flux question)	A/FV
In the direction of $\vec{B}$ (area vector $\vec{A}$ in the flux question)	B/AV
In a direction between $\vec{A}$ and $\vec{B}$	BT
In the direction perpendicular to both $\vec{A}$ and $\vec{B}$ and into the page/screen	IP
In the direction perpendicular to both $\vec{A}$ and $\vec{B}$ and out of the page/screen	OP
Clockwise	CW
Counterclockwise	CCW
Not enough information to tell	NEI

The first two items used mathematical contexts. Students were provided with diagrams containing two vectors in space connected tail-to-tail, with labels $\vec{A}$ and $\vec{B}$. The vectors and labels were the only information provided in item 1 (Figure 1(a)); item 2 included a Cartesian coordinate grid and noted angles of each vector with respect to the positive *x* axis (Figure 1(b)). Item 1 asked about only the directions of the vector products $\vec{A}\cdot\vec{B}$ and $\vec{A}\times\vec{B}$, and subsequent items asked about the direction and magnitudes of the vector products.

Item 3 used the physical context of electric or magnetic flux through a surface area (Figures 1(c) and 1(d)); the choice of field was course dependent. In either case the flux due to a uniform field through a planar surface is given by a dot product of the field and surface area vectors (e.g., $\Phi_E = \vec{E}\cdot\vec{A}$). The potential for both positive and negative flux through a surface can sometimes be conflated for flux having a direction rather than being indicative of source or sink of field.

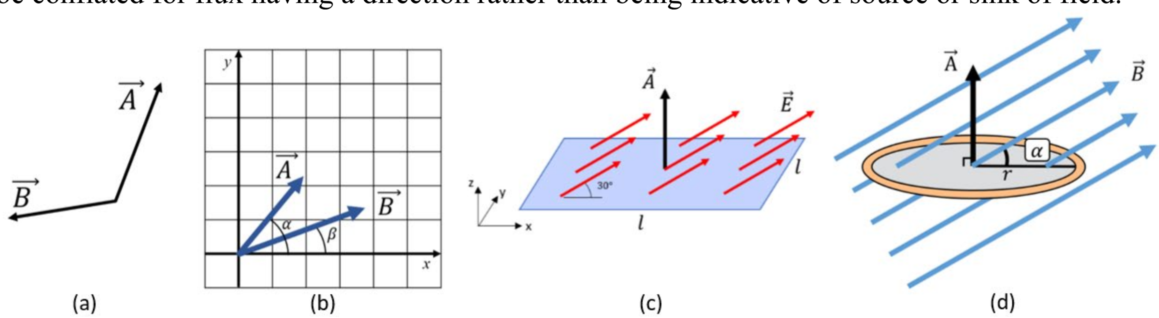


Figure 1. Diagrams provided in survey items asking about the direction of the dot product or the flux for the given situation. Diagrams for (a) item 1, (b) item 2, (c) item 3 for electric flux, and (d) item 3 for magnetic flux.

The survey was administered at a medium-sized public US university in three different physics courses: second-semester introductory calculus-based physics (N=153), sophomore-level mathematical methods in physics (math methods; N=14), and the second semester of junior-level electricity and magnetism (E&M; N=6). In each course, the survey was administered after instruction on the relevant mathematics or physics context(s). The context for item 3 for each course was most relevant to the context of instruction: magnetic flux in the introductory and E&M courses and electric flux in the math methods course.

To address the question of how student understanding of vector dot products differs at different levels of the physics curriculum we conducted a pseudo-longitudinal comparison between the three courses. For this comparison we limit the pool of introductory students to the physics and engineering physics majors who will go on to take the upper level courses (N=9).

Results

Performance across the three items related to dot products is presented for the introductory students before comparing the performance of students at different levels.

Introductory level

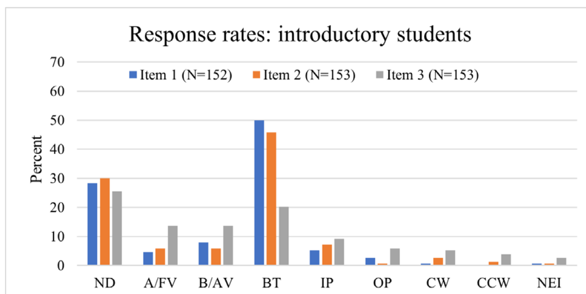


Figure 2. Introductory-level student performance on items 1, 2 and 3. See table 1 for answer choice descriptions.

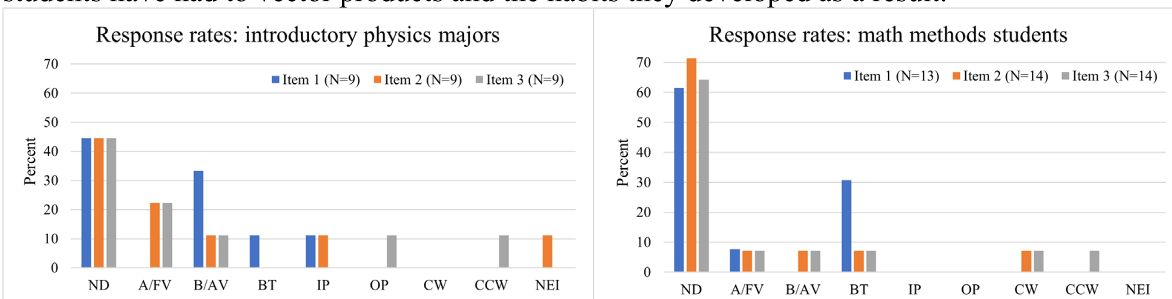
All three items had a similar percentage of introductory students selecting the correct response (Figure 2): 28% for item 1, 30% for item 2, and 25% for item 3. The most common choice for both items 1 and 2, 50% and 46% respectively, is that the direction of the dot product is between the two vectors (BT). This is consistent with the most common response seen by Van Deventer in a question asking which arrow, if any, describes the direction of the dot product of two given vectors (Van Deventer, 2008). Item 3, the flux question, had the most evenly distributed answer choices, suggesting less consistent understanding among students.

Similar percentages of introductory students selected the correct choice in each context, however only 16% of the introductory population selected the correct choice across all three items. Another 14% of students said the direction of the dot product is between the two vectors on all three items and 3% of students said that the direction of the dot product is in the direction of one of the two vectors (selecting either A/FV or B/AV) for all three questions. The remaining 67% of students did not answer consistently across all three contexts, which suggests that they do not have a coherent understanding of the dot product in multiple contexts.

Pseudo-longitudinal

The correct answer, "it does not have a direction," was the most common response on all questions through all three levels. At the introductory level, 44% of physics majors selected the correct answer on each item (Figure 3(a)). Between 62% and 71% of the math methods students chose the correct answer for each item (Figure 3(b)), and as expected, the junior E&M students had the highest performance of the three populations, with 100% correct in all three cases.

In all three levels, performance on the flux question, item 3, was similar to the performance on mathematics context questions, consistent with the similarities between isomorphic mathematics and physics questions in which mechanical work was used as the physics context conducted by Carli et al. (2020) and Van Deventer (2008). The most common incorrect answer in item 1 is different between the two lower courses: introductory students predominantly selected that the dot product is in the direction of $\vec{B}$, while math methods students selected that the dot product is between $\vec{A}$ and $\vec{B}$. The type of error may depend on how much exposure students have had to vector products and the habits they developed as a result.



(a) *(b)*

Figure 3. Performance by physics majors in (a) introductory and (b) math methods courses on items 1, 2 and 3. See Table 1 for answer choice descriptions.

Discussion and Conclusion

The performance of introductory students in this preliminary study (25-30%) is similar to end-of-semester introductory student performance on dot product direction questions in Van Deventer's study, with 25% of students in a mathematics context and 24% of students in a physics context selecting no direction (Van Deventer, 2008). The most common incorrect answer, a direction between $\vec{A}$ and $\vec{B}$, could be based on properties of vector addition. Some student explanations explicitly reference the tip-to-tail method, which is a strategy for vector addition not multiplication. For the introductory students, there seemed to be a wider variety of responses to the flux question than to the mathematics context questions. Asking these questions in other physics contexts (e.g., mechanical work) could document the relative difficulty of flux as a physics context for considering the dot product direction.

The trend of improved results as the course level increases is expected in the pseudo-longitudinal study, but the performance is not as good as one might expect, especially in the sophomore math methods course. Vectors and vector operations are introduced in the introductory sequence and proficiency in subsequent courses is almost assumed. A 100% correct response rate on all questions in junior E&M is a welcome result, but to promote student understanding of physics concepts in intermediate and upper division courses, vector operations need to be second nature as soon as possible, not after over half of the physics curriculum. While we see that student performance improves as we look at higher-level courses, a limitation of this study is the small sample size in each course.

Future work includes collecting more survey data, both internally and at other institutions, conducting interviews to elicit more detailed explanations of student reasoning, and connecting empirical data to existing frameworks.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. PHY-1912087. We are grateful to A. Piña for assistance with manuscript preparation.

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