Isomorphism across Courses: Students' Metaphors in Discrete Math, Linear Algebra, and Abstract Algebra

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Isomorphism appears across many courses in mathematics, including discrete mathematics, linear algebra, and abstract algebra. However, examinations of student understandings of isomorphism have mostly focused on group isomorphism in abstract algebra. In this paper, we compare survey responses from 49 discrete math, 19 linear algebra, and 27 abstract algebra students who were prompted to explain the concept of isomorphism to a child. Results include a cross-course emphasis on types of sameness but variations in the types of language used to characterize isomorphism in each course. Implications include the need for care among instructors and researchers when referring the concept of "isomorphism" without a context as well as the need for further work in understandings of graph theory.

Keywords: isomorphism, abstract algebra, linear algebra, discrete math

Introduction and Background Literature

Isomorphism, especially group isomorphism, in abstract algebra has received extensive attention in the RUME community. Group isomorphism was among the first topics studied in RUME (e.g., Dubinsky et al., 1994; Leron et al., 1995), and has received sustained attention for its connections to proof construction (e.g., Weber, 2001; Weber & Alcock, 2004), curriculum design (e.g., Larsen, 2013), understanding functions (e.g., Melhuish et al., 2020; Rupnow, 2017), and conceptions of sameness (e.g., Rupnow et al., 2022; Rupnow & Sassman, 2022). This focus is understandable, given its centrality to the abstract algebra curriculum (Melhuish, 2015).

However, isomorphism has not received comparable emphasis in discrete math or linear algebra. Like in abstract algebra, graph isomorphism is central to understanding graph theory and, more broadly, discrete math (Ebert et al., 2004). However, most research on discrete math understandings have focused on other areas, such as number theory (e.g., Zazkis & Campbell, 1996; Zazkis & Liljedahl, 2004) and combinatorics (e.g., Lockwood, 2013; Lockwood & Gibson, 2016). Research in linear algebra has also focused on other areas, such as bases (Serbin et al., 2021; Stewart & Thomas, 2010; Wawro et al., 2012), eigentheory (e.g., Serbin et al., 2020; Thomas & Stewart, 2011), and linear transformations (e.g., Andrews-Larson et al., 2017), though change of basis and linear transformations are relevant to understanding isomorphism in linear algebra. Moreover, comparative examinations of concepts appearing in multiple areas have focused on broad topics like functions (e.g., Oehrtman et al., 2008) and sameness (e.g., Rupnow et al., 2022) rather than concepts using the same term but with subtly different meanings in different contexts like isomorphism. Thus, we address two research questions:

1. How are characterizations of isomorphism similar across discrete mathematics, linear algebra, and abstract algebra students?
2. How are characterizations of isomorphism different across discrete mathematics, linear algebra, and abstract algebra students?

Theoretical Perspective

Conceptual metaphors is a theoretical lens that illuminates how individuals' thinking is structured based on their language choices (e.g., Lakoff & Johnson, 1980; Lakoff & Núñez, 1997). Cross-domain conceptual mappings relate the cognitive structure of a target concept (e.g., isomorphism) to the more developed thoughts in source domains (e.g., same properties, transformation). For instance, "An isomorphism is a transformation" is a conceptual metaphor that gives information about a target domain (isomorphism) by relating it to a source domain that is already understood in some way (a transformation).

Previous work using conceptual metaphors include analyses of mathematicians' views of isomorphism and homomorphism in abstract algebra (Rupnow, 2021; Rupnow & Randazzo, 2022; Rupnow & Sassman, 2022). Other work has examined undergraduate students' understandings of various concepts, including bases and linear transformations in linear algebra (Zandieh et al., 2017; Adiredja & Zandieh, 2020) and isomorphism and homomorphism in abstract algebra (Melhuish et al., 2020; Rupnow, 2017). Here we extend the conceptual metaphors for isomorphism in abstract algebra identified in Rupnow (2021) and built upon in Rupnow and Randazzo (2022) to parallel uses in linear algebra and discrete mathematics.

Methods

Data was collected from surveys sent to six groups of students: two sections each of the first author's discrete mathematics (DM) and linear algebra (LA) courses and two sections of other instructors' abstract algebra (AA) courses. DM students were mostly computer science majors, LA students were STEM majors (including computer science, engineering, mathematics, and physics majors), and AA students were math majors or minors. The surveys were distributed after students had learned about isomorphism in all sections. Initial questions on the discrete math and linear algebra surveys asked about sameness in mathematics whereas initial questions on the abstract algebra survey assessed beliefs about learning (especially related to fixed or growth mindset). Here we analyze responses to one question from the middle of the survey: "How would you describe an "isomorphism" to a ten-year-old child?" (AA) or "How would you describe the concept of isomorphic/isomorphism to a ten-year-old?" (DM/LA). 21 and 28 students completed the DM survey in the two sections; 9 and 10 students completed the LA survey in the two sections; and 13 and 14 students completed the AA survey in the two sections. We did not observe major differences between the sections for each course and thus combine our reporting into a dataset of 49 DM responses, 19 LA responses, and 27 AA responses.

Data analysis was conducted by two researchers who coded independently, then discussed and came to consensus for each response. We recognize students were prompted to provide comparative explanations based on the task of explaining in a manner understandable by a child. Nevertheless, our goal was to classify the underlying meanings conveyed by these explanations according to our interpretations of the participants' responses. As such, we used the metaphors for isomorphism in abstract algebra in Rupnow (2021) as an initial codebook, adding new metaphors as necessary. This process aligns with codebook thematic analysis (Braun et al., 2019), in which existing codes are used as a basis for coding but new codes are permitted to be added to adequately capture the nuances in data.

Results

We organize the results of our coding into three subsections, each corresponding to a different course. We also provide Table 1, which offers an initial overview of the frequencies of each metaphor by course. Metaphors are organized by cluster, with some belonging to sameness-based clusters such as sameness (*generic sameness*, *same properties*, *same but looks different*, *other branch analogues*), sameness/mapping (*renaming/relabeling*, *matching*), and sameness/formal definition (*structure-preservation, operation-preservation*) clusters. Other metaphors belong to the mapping cluster (*generic mapping/relation*, *morphing/transformation*, *invertible, journey*), the formal definition cluster (*literal formal definition*), or made *no attempt*/*unclear*.

Table 1. Frequencies of metaphors by course.

Metaphor Cluster	Metaphor Code	Discrete Math (n = 49)	Linear Algebra (n = 19)	Abstract Algebra (n = 27)
Sameness	Same but looks different	19	7	3
	Same properties	19	0	8
	Generic sameness	9	3	14
	Other branch analogues	0	0	1
Sameness/ mapping	Matching	4	4	3
	Renaming/relabeling	1	0	2
Sameness/formal definition	Structure-preservation	2	0	0
	Operation-preservation	0	1	0
Mapping	Morphing/transformation	2	9	2
	Invertible	4	2	0
	Generic mapping	1	0	5
	Journey	0	1	1
Formal definition	Literal formal definition	0	0	1
No attempt/ unclear	Unclear/no attempt	5	0	1

Discrete Math

We begin with metaphors used by Discrete Math (DM) students to describe isomorphism. The DM textbook defined isomorphism as: “A graph G_1 is isomorphic to a graph G_2 when there is a one-to-one correspondence f between the vertices of G_1 and G_2 such that the vertices U and W are adjacent in G_1 if and only if the vertices $f(U)$ and $f(W)$ are adjacent in G_2. The function f is called an isomorphism of G_1 with G_2” (Dossey et al., 2005, p. 159). Both *same properties* and *same but looks different* were commonly used metaphors, as was *generic sameness*.

Many of the metaphors discussed shared properties across groups of objects. As isomorphism in DM is centered on graphs, a large number of these metaphors identified the same properties across different graphs: “look and see if the two shapes of the graphs are identical, vertices, edges, weights.” Other responses emphasized common properties across everyday objects, such as identifying similarities among houses or vehicles.

Another commonly used metaphor used to describe isomorphism was *same but looks different*. These types of instances place emphasis on the idea that while isomorphic objects have the same underlying structure or properties, they may appear different visually. The context of these metaphors varied; as with *same properties*, many discussed graphs: “we might have two different graphs…and even though they might look different, if we can rearrange the points and lines in one of the graphs to make it look exactly like the other graph, then we can say that the two graphs are isomorphic.” Others emphasized the underlying action behind these surface differences through a variety of mediums, including sticks and puzzles, by demonstrating that physical manipulation does not change the inherent identity of the object being acted upon.

Still another metaphor characterizing isomorphism was *generic sameness.* The majority of metaphors in this category emphasized the underlying sameness of two graphs: “It’s when something has a similar form or shape.” In addition to more commonly utilized metaphors, *structure-preservation* was unique to DM in our data set. Two students chose to emphasize the *structure-preservation* property of isomorphism: “an isomorphism is a structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping.”

Linear Algebra

We now move our focus to Linear Algebra (LA) students’ depictions of isomorphism. The LA textbook’s definition for isomorphism was: “Let V be a real vector space with operations $\oplus$ and $\odot$, and let W be a real vector space with operations $\boxplus$ and $\boxdot$. A one-to-one function L mapping V onto W is called an isomorphism (from the Greek *isos*, meaning “the same,” and *morphos*, meaning “structure”) of V onto W if (a) $L(\mathbf{v}\oplus\mathbf{w}) = L(\mathbf{v}) \boxplus L(\mathbf{w})$ for $\mathbf{v}$, $\mathbf{w}$ in V; (b) $L(c \odot \mathbf{v}) = c \boxdot L(\mathbf{v})$ for $\mathbf{v}$ in V, c a real number. In this case we say that V is isomorphic to W.” (Kolman & Hill, 2007, p. 258). Among LA students, the most common metaphors were *morphing/transformation*, *same but looks different*, and *matching*.

Metaphors describing the act of *morphing* or *transformation* were the most common type used. Many such instances emphasized the ability of isomorphic objects to be physically manipulated into one another: “You can take one shape, move it, stretch it, or squish it into another shape, and be able to change it back to the original.” This response also notes the *invertibility* of such transformations. Still others highlighted the underlying shared structure of isomorphic objects even under a transformation: “When you have a group of things and make the same change to everything in that group that is an isomorphism. So if I had a bucket of crayons and then shaved the tips off of all of them they would be isomorphic before and after.”

Same but looks different instances emphasized that although an object or group of objects may look different, they still share core underlying properties: “I would describe isomorphism through the use of tiles or legos with numbers on it. The idea being that no matter how you rearrange the pieces, all of the numbers are still there, and you can trace the method used [to] mix up the shape.”

Several students also discussed isomorphism in terms of *matching*. These examples often invoked non-mathematical objects such as dice or people, and emphasized an underlying correspondence or bijectivity:

> If we have 10 classmates on one side of a classroom, and 10 classmates on the other side of a classroom. If all 10 classmates went to the other side of the room, and found a friend. It is isomorphic if everyone has only one friend, and everyone has a friend.

In our data, Linear Algebra had the only use of *operation-preservation*: “if you were to change something in group one by adding or multiplying so there is more of it, there must be the same increase in the transformed group 2.”

Abstract Algebra

Finally, we examine Abstract Algebra (AA) students’ isomorphism descriptions. The two sections of AA used different textbooks. In one, the textbook’s definition for isomorphism was: “Let G_1 and G_2 be groups. A bijective function f: $G_1 \to G_2$ with the property that for any two elements a and b in G_1, $f(ab) = f(a)f(b)$ is called an isomorphism from G_1 to G_2. If there exists an isomorphism from G_1 to G_2, we say that G_1 is isomorphic to G_2 (Pinter, 2010, p. 94).” In the other section, the definition given was: “Two groups $(G, \cdot)$ and $(H, \circ)$ are isomorphic if there exists a one-to-one and onto map ϕ: $G \to H$ such that the group operation is preserved; that is,

$\phi(a \cdot b) = \phi(a) \circ \phi(b)$ for all a, b in G. If G is isomorphic to H, we write $G \cong H$. The map ϕ is called an isomorphism (Judson 2019, p. 119)." Among these students, *generic sameness* and *same properties* were most frequently utilized, along with *generic mapping*.

Depictions invoking *generic sameness* were the most prominent among AA students, emphasizing some flavor of sameness that unites isomorphic objects: "It is when 2 things are similar in every way and act the same way when you do something to it." Also commonly utilized was *same properties*, which stresses specific shared properties across isomorphic objects: "Draw a group of triangles and a group of squares that have the same number of elements. The groups are similar because they are both shapes, or are of the same color, etc.". Several responses utilized both metaphors, such as the following which emphasizes both specific traits and generalized sameness: "With two bunches of objects, if they have the same number of things and each thing's traits are similar to one in the other bunch, they're probably isomorphic."

Generic mapping instances focused on the presence of some mapping between isomorphic objects: "It is a map that shows that two different things are the same." Unique to AA in our data were *other branch analogues* and *literal formal definition*, each used by a single student. *Other branch analogues* identified a parallel between isomorphism and equality: "It is similar to an equals sign except it can be between groups of numbers not just equations." An example of *literal formal definition* emphasized how elements are required to interact under isomorphism:

> When you have 2 groups A and B and all of the rules between the groups A and B are followed. There are also additional rules that A and B must follow. They must when given a number to plug in give you a number back, and if you put in the two numbers that are different they must give you different numbers back. For example if you put in the number 2 to A and put in the number 3 for A and get back 4 for both then it does not follow the rules. The same rules apply to group B.

Discussion

This study examined students' characterization of isomorphism in Discrete Math (DM), Linear Algebra (LA), and Abstract Algebra (AA) courses. Of note, each course had a different most-common metaphor. DM tied between *same properties* and *same but looks different*; LA's most common metaphor was *morphing/transformation*; and AA's most common was *generic sameness*. From this we note that while "isomorphism" is a shared name for concepts across courses, students still chose subtly different things to emphasize, indicating the need for care when referring to isomorphism and that isomorphism can mean different things in different contexts. This aligns with prior work with mathematicians, who carefully attended to the context-dependent nature of isomorphism (Rupnow & Sassman, 2022).

Despite these differences in emphasis, we note that metaphors from sameness-based clusters dominated the DM and AA students' descriptions of isomorphism and were a second emphasis behind the mapping cluster in LA. These differences do not appear to be related to the textbook's definition of the concepts given, as all definitions emphasized the function/mapping nature of isomorphism and the definition that directly used the word "same" was from LA, whose students had the least emphasis on sameness. However, the broader course contexts may have been impactful in terms of the objects to which isomorphism is applied, broader emphases of the course (i.e., proof), and whether isomorphism had been encountered in a previous course. Future research should compare the broader instructional contexts to better understand how students' perceptions of isomorphism were formed and what proved most impactful.

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