Teaching Proofs in a Second Linear Algebra Course: A Mathematician's Resources, Orientations, Goals, and Continual Decision Makings

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In this research study, we employed Schoenfeld's (2010) theory of Resources, Orientations, and Goals (ROGs) to examine an instructor's (and co-author's) textbook in a second course in linear algebra. The course was proof-based and the instructor anticipated that many students would have difficulty. Having the knowledge of students' difficulties from a first course in linear algebra they created tailored resources with many hints and pedagogical attributes for their students. The instructor's goal was to encourage students to construct their own proofs to understand and learn the materials, hence promoting exploration, conjecturing, and proving mental mathematical actions.

Keywords: linear algebra, proof, second course, belief, pedagogy, knowledge of content and students, mathematical knowledge for teaching

Literature Review

Linear algebra is a key topic for many mathematics majors and other fields. The Linear Algebra Curriculum Study Group (LACSG) recommended that "at least one second course in matrix theory/linear algebra should be a high priority for every mathematics curriculum" (Carlson, Johnson, Lay, & Porter, 1993, p. 45). The LACSG 2.0 recommends that mathematics departments offer a variety of second courses (e.g., numerical linear algebra) and include wider topics (Stewart et al., 2022). However, research on topics in second courses of linear algebra, which contain more abstract content, is rare. In addition, in a survey paper by Stewart, Andrews-Larson, and Zandieh (2019), the authors found that more research on how students make sense of linear algebra proofs are needed. Hence, in this paper, we focus our attention on linear algebra proofs. For example, Stewart and Thomas (2019) aimed to uncover linear algebra students' perceptions of proofs in a first course. The results revealed that many students expressed their need for understanding. Studies also showed that the number of new definitions which linear algebra students must learn to begin writing proofs is overwhelming and makes learning proofs difficult (Hannah, 2017; Britton & Henderson, 2009). Malek and Movshovitz-Hadar (2011) employed one-on-one workshops to examine the effect of using their Transparent Pseudo Proofs (TPPs) in teaching first-year linear algebra proofs. They found for non-algorithmic proofs, students with familiarities in the TPPs wrote more in-depth and satisfactory answers than students who learned proofs traditionally, however, both groups of students performed equally for algorithmic proofs. Uhlig (2002) developed a novel approach compared to the traditional Definition, Lemma, Proof, Theorem, Proof, Corollary (DLPTPC) to teach linear algebra proofs. He posed the following questions: "What happens if? Why does it happen? How do different cases occur? What is true here?" (p. 338). He believed "Such a WWHWT sequence of presentation quickly leads students to understand, construct, reason through, enjoy, and actually demand 'salient point' type proofs" (p. 338).

Further expanding on the ideas of proof education, Melhuish et al. (2022) synthesized 104 papers from a range of countries and methods to describe what is known about proof-based learning and what is still missing. In their exploration, they uncovered that two schools of thought existed for teaching proofs, lecture-based proofs and student-centered learning. They concluded that while student-based learning is promising, a large gap remained in "understanding of the theoretical mechanisms which specific instructor moves may encourage or scaffold student activity is only beginning to emerge" (Melhuish et al., 2022, p. 17). Attempting to understand the theoretical mechanisms of an instructor is not something that is entirely new despite not being well explored. Using Schoenfeld's (2010) Resources, Orientations, and Goals (ROGs) framework, Hannah, Stewart, and Thomas (2011) attempted to describe the instructors' overarching goals to present linear algebra topics to their students. The paper found that analyzing the instructor's goals proved useful in uncovering his decision making and how the instructor approached teaching the "big picture." These results validated the instructor's decision making as a useful tool for improving the teaching ability of the instructor. Furthermore, the feedback from the students in this case of reflective teaching was positive, reinforcing the idea that this analysis provided enrichment for the instructor and students.

Theoretical Framework

The theoretical aspects of this study are based on Schoenfeld's (2010) Resources, Orientations and Goals (ROGs). He claims that "if you know enough about a teacher's knowledge, goals and beliefs, you can explain every decision that he or she makes, in the midst of teaching" (2012, p. 343). By resources Schoenfeld focuses mainly on knowledge, which they define "as the information that he or she has potentially available to bring to bear in order to solve problems, achieve goals, or perform other such tasks" (2010, p. 25). Goals are defined as what the individual wants to achieve. The term orientations refer to "dispositions, beliefs, values, tastes, and preferences" (2010, p. 29). Although, the theory was originally applied on school teaching, (e.g., Aguirre & Speer, 2000; Thomas & Yoon, 2011; Törner, Rolke, Rösken, & Sriraman, 2010), it also has applicability to teaching at university (e.g., Hannah, Stewart & Thomas, 2011; Paterson, Thomas & Taylor, 2011; Stewart, Troup, & Plaxco, 2018). In addition, as we are interested in understanding the teaching and learning of proof in a formal second linear algebra course, we approach this problem with the Mathematical Knowledge for Teaching framework (MKT) by Ball, Thames, and Phelps (2008). This framework identifies several distinct domains of knowledge that instructors must use when they teach mathematical content.

In particular, for this paper, we restrict ourselves to one domain: *knowledge of content and students (KCS).* According to Ball et al., (2008, p. 401):

> *[K]nowledge of content and students (KCS)*, is knowledge that combines knowing about students and knowing about mathematics. Teachers must anticipate what students are likely to think and what they will find confusing. When choosing an example, teachers need to predict what students will find interesting and motivating. When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. … Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking.

In this study, our goal is to network these theories by associating Ball's instructor anticipation KCS and Schoenfeld's instructor beliefs about student resources. The research question guiding this research is: What knowledge of content and students do experienced instructors leverage for teaching proofs in linear algebra?

Methods

This narrative study (Creswell, 2013) is part of a larger study working with a mathematician (the instructor and first author), whose research is in geometry and algebra, and his linear algebra students. In the rest of the paper, we shall refer to the mathematician as "the instructor." The research team consisted of the instructor, a mathematics education researcher in linear algebra education, and a graduate student who is working on linear algebra education. The course was self-contained, proof-based, and constructed the theory of formal linear algebra from basic set-theoretic assumptions. Incoming students all had experience with formal proofs in other courses. The author approached this course with the core pedagogical perspective that *students should have the opportunity to construct all proofs themselves*.

Students were supplied with a self-contained textbook written by the instructor that had all definitions, propositions, and reflections, carefully sequenced, that students would progress through during the entire term. Recognizing the difficulty of creating proofs on their own, the instructor supplied *scaffolding* (via hints, questions, and reflections) to support the students in the completion of the proofs without reducing the cognitive demand of the proofs themselves.

Class met twice a week for 75 minutes. Class time was split into three components: lecture (~15min), group time (~45min), and presentations (~15min). Throughout lecture, the instructor would recall concepts from the past class and introduce new ideas for that day's investigations. During group time, groups would work, while standing up at and writing on white boards, to construct (and convince one another) of proofs to that day's propositions. During this time, the author would walk among the groups, attempting to facilitate discussions and provide hints for groups that were stuck. Towards the end of this time, the author would recruit groups to share their thinking for their proofs. During presentations, groups would present their proofs (complete or incomplete) to the class for discussion.

Each week, students were then required to LaTeX up and submit all proofs in their "course journal," a single cumulative document. Students were not allowed to look up proofs from other resources, but they were encouraged to work with classmates outside of class.

Data collection and analysis

We collected data from the textbook created by the instructor, videos of the instructor during lecture, videos of the students presenting, videos of student board work, and the final course journal submissions for each student. In this paper we only analyzed the instructor's textbook. The team wanted to look at a small sample and identify structures of linear algebra proofs. The members of the team collectively decided to analyze Investigation 12 (Linear Combinations and Span), both because it related to previous work (Madden et al 2023) and the team determined it was the first instance of propositions which contained sophisticated linear algebraic reasoning. The team then determined it was beneficial to also analyze Investigation 9 (Subspaces), as Investigation 12 frequently referenced it. The research team individually analyzed the data, identified emergent themes, and then the team met to share, discuss, and settle upon identified themes. The team was especially interested in the hints. The team continually consulted with the instructor to gain insights into his knowledge and beliefs that led him to make the decisions that were observed in the investigations. In this paper, we will only analyze the instructor's beliefs.

Results and Discussion

We analyzed two investigations on subspaces and span that the instructor provided his students (see Figures 2, 3, and 4). These investigations, as well as relevant references to earlier investigations, are provided below (see also Figure 1). These investigations were created by the instructor prior to his students using them. As such, these investigations contain a significant amount of information about the instructor's knowledge and beliefs about the content, his students, and teaching the subject, as well as his goals.

Instructor's Knowledge and Beliefs About Learning Proof in a Student-Centric Course

The instructor has taught a proof-based second course in linear algebra many times, and based upon those experiences, wrote his own self-contained textbook that he based his course upon and that he distributed to his students. As a consequence of his experiences teaching such a course, together with his beliefs concerning how people learn, the instructor came to hold certain beliefs regarding the structure of such a second course, and which caused him to make certain teaching decisions regarding the nature and structure of his textbook. Here are some of the instructor's beliefs and corresponding teaching decisions.

Belief (B1). Students should learn formal linear algebra in a student-centric course, which would include that the students should have the opportunity to construct all proofs themselves. Belief B1 resulted in the teaching decision (TD1): he created a self-contained textbook that contained no proofs, but rather a collection of carefully sequenced propositions and reflections.

Belief (B2). Teachers play an important role in student learning by (1) providing students with cognitively demanding problems and (2) helping students access knowledge and heuristics to solve the problem. In this context, the problems provided were proofs and Belief B2 resulted in teaching decision (TD2): for each proof, anticipating where students would struggle, he carefully wrote hints, which typically included a suggestion for the type of proof to pursue, the definitions to retrieve, and some intermediate results to prove. This can be observed in Figure 4 in the hints to the proofs of Propositions 4.48 and 4.50.

Belief (B3). Central to the story of a proof-based second course in linear algebra is (1) a solid understanding of linear algebra for the object $\mathbf{R}^n$, and (2) continual enactment of structural abstraction (Tall, 2013). Belief B3 resulted in the teaching decision (TD3): prior to a (formal) object being constructed, the antecedent object in $\mathbf{R}^n$ was revisited and discussed, as was explicitly the action of structural abstraction. The terminology of structural abstraction was introduced early in the textbook (see Figure 1) and then continually used throughout the term.

> A fundamental mathematical action is **structural abstraction**, in which we take a *specific* object within a *specific* context, observe its meaningful structures, and then construct an abstract object endowed with these structures, no more and no less. We then may mentally act upon this new abstract object, bringing it to life as a new mathematical entity, of which our original object is now a mere example.

Figure 1. Excerpt from Investigation 3 of the instructor's textbook.

Teaching decision 3 (TD3) can be seen having been implemented in the introduction to subspaces (see Figure 2).

> **Investigation 9**. **Subspaces.** Once again, think back to your first course in linear algebra. We were constantly looking at spans and solving matrix equations. Geometrically, we were looking it, lines, planes, and more generally, k-planes, through the origin. We now perform a structural abstraction on these objects, which we now formalize with subspaces.
>
> **Definition 4.24.** If V is a vector space over the field F, then a **vector subspace** of V is a subset $W \subset V$ satisfying:
>
> **(S1)** W is nonempty,
> **(S2)** W closed under addition: if $\mathbf{w}_1, \mathbf{w}_2 \in W$, then $\mathbf{w}_1 + \mathbf{w}_2 \in W$,
> **(S3)** W is closed under scaling: if $a \in F$ and $\mathbf{w} \in W$, then $a\mathbf{w} \in W$.
>
> I will often use the symbol W for a subspace.

Figure 2. Excerpt from Investigation 9 of the instructor's textbook.

Instructor's Knowledge and Beliefs About the Learning of Mathematical Mental Actions

Belief (B4). There are many distinct types of mathematical mental actions, including proving, conjecturing, generalizing, abstracting, modeling, and interpreting (Harel, 2008). It is a goal of the instructor that in a formal course such as this, students internalize these actions.

Beliefs (B5-B7) of the instructor is that students (1) come into such a course not being explicitly aware of such mental actions, (2) students are more likely to internalize such actions if these actions are explicitly named and discussed, and (3) students are more likely to internalize such actions if they have opportunities to engage in them.

Reflection 4.30. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Carefully explain your thinking.

Proposition 4.31. *If F is a field, $n \in \mathbb{Z}_+$, and $i \in \{1,2,\dots,n\}$, then the set* $W_i := \left\{ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in F^n \;\middle|\; v_i = 0 \right\}$ *is a vector subspace of F^n.*

Generalization is another important mathematical action we perform, distinct from, but related to structural abstraction. See, for example, the work of Harel and Tall, The General, the Abstract, and the Generic in Advanced Mathematics. Roughly speaking, **expansive generalization** occurs when you *expand the applicability* of an object or process with which you are already familiar.

Reflection 4.32. A common example of expansive generalization is given by the movement from linear combinations of vectors in $\mathbb{R}^2$ to linear combination of vectors in $\mathbb{R}^n$. Carefully explain the process of finding linear combinations in $\mathbb{R}^2$. Then, explain how this familiar process is expanded to $\mathbb{R}^3$, and eventually, $\mathbb{R}^n$. Explain how this fits the criteria for expansive generalization.

Reflection 4.33. Compare and contrast expansive generalization with structural abstraction. You may find it useful to use as examples, $\mathbb{R}^2 \to \mathbb{R}^n$ for expansive generalization and $\mathbb{R}^n$ to an abstract vector space V for structural abstraction.

Explore-Conjecture-Prove 4.34. Using expansive generalization upon the subspaces $W_i \subset F^n$ in Proposition 4.31, find, clearly conjecture, and then verify a more general collection of subspaces of F^n.

Figure 3. Excerpt from Investigation 9 of the instructor's textbook.

The course is built around proving, however, the instructor identified and highlighted other mental actions where he could. In this instance (Figure 3), the instructor made the teaching decision (TD4) to have students reflect upon and practice generalizing and conjecturing. This sequence comes at the end of the initial investigation into subspaces. Here, in Proposition 4.31, students are initially asked to prove that the subset W_i of F^n, where the ith entry is zero, is a subspace. The instructor provided a hyperlink to a mathematics education resource to reflect upon expansive generalization by Harel and Tall (1989) and then asked students to use expansive generalization to conjecture and then prove new propositions (see Figure 3). More precisely, students are asked to reflect upon the method in which they proved Proposition 4.31, and then in Explore-Conjecture-Prove 4.34 to determine other subsets that are subspaces. Here students were free to consider many generalizations, the two most obvious being: (1) if the ith entry is held a different (nonzero) constant (which will not be a subspace) or (2) if more than one entry is held to be zero (which is a subspace).

Investigation 12. **Linear Combinations and Span.** One core idea of linear algebra is *mixture*. Vector space axioms were were precisely the structures necessary to allow for mixtures, which we describe formally as linear combinations.

Throughout this investigation, V will denote a vector space over the field F.

Definition 4.45. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$, then a **linear combination** of S is an expression of the form

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n,$$

where $x_1, x_2, \ldots, x_n \in F$.

Note that we only take *finite sums* in linear combinations.

Definition 4.46. The **trivial** linear combination is the one with all 0 scalars. Otherwise it is **nontrivial**.

Definition 4.47. The **span** of S, denoted $\mathrm{span}(S)$, is the set of all linear combinations of vectors in S. (Note that S can be infinite, but the combinations are finite.) In symbols,

$$\mathrm{span}(S) := \{x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \mid \mathbf{v}_1, \ldots, \mathbf{v}_n \in S \text{ and } x_1, x_2, \ldots, x_n \in F\}.$$

Sometimes if $\mathbf{v} \in V$, then I will write $\mathrm{span}(\{\mathbf{v}\})$ as $F\mathbf{v}$ which reminds us that the span of a single vector is just the line created by all possible scalings of that vector.

Proposition 4.48. *If $S \subset V$ is nonempty, then* $\mathrm{span}(S)$ *is a vector subspace.*
(Hint: Try a direct proof. Show that $\mathrm{span}(S)$ satisfies (S1)-(S3). To show (S2), you will need to show that, if $\mathbf{w}_1, \mathbf{w}_2 \in \mathrm{span}(S)$, then $\mathbf{w}_1 + \mathbf{w}_2 \in \mathrm{span}(S)$. To do this, you need to interpret and unravel definitions. Similar for (S3). You may find Definition 4.47, Definition 4.45, Definition 4.24, and Definition 4.1 useful.)

A useful alternative characterization of span is that it is the *smallest* subspace containing S, which is made precise by the following proposition.

Lemma 4.49. *If $S \subset V$ is nonempty, $W \subset V$ is a subspace, and $S \subset W$, then* $\mathrm{span}(S) \subset W$.
(Hint: Try a direct proof. You need to show that any linear combination of vectors in S is contained within W. You may find Definition 4.47, Definition 4.45 and Definition 4.24 useful.)

Proposition 4.50. *If $S \subset V$ is nonempty, then* $\mathrm{span}(S)$ *is equal to the intersection of all subspaces of V which contain S.*
(Hint: Try a direct proof. Let $\mathscr{A}$ denote the collection of all subspaces of V containing S. In symbols, $\mathscr{A} = \{W \subset V \mid W \text{ is a subspace and } S \subset W\}$. Then you want to show $\mathrm{span}(S) = \cap_{W\in\mathscr{A}} W$. You will need to show both inclusions. To show that $\cap_{W\in\mathscr{S}} W \subset \mathrm{span}(S)$ is immediate (why???). To show the other inclusion, observe S is in each W for which $W \in \mathscr{A}$, and deduce $\mathrm{span}(S) \subset W$ (how???). You may find Proposition 4.48, Lemma 4.49, and Definition 2.5 useful.)

Figure 4. Excerpt from Investigation 12 of the instructor's textbook.

Instructor's Knowledge and Beliefs About the Learning of Span

The instructor identified two characterizations of the concept of the span of a set of vectors S, span(S), as we now outline.

The first conception of span is a "bottom-up" construction. One first conceives of a linear combination. Based on the instructor's APOS (Dubinsky & McDonald, 2001) theoretical knowledge, the action of computing an explicit linear combination of vectors in $\mathbf{R}^n$ for small n, is a task that is introduced early in a first course in linear algebra. Through repeating this action, a student may internalize this action to create an internalized procedure. Eventually the student may encapsulate this procedure as a single object on its own. Only once a student has internalized the notion of a single linear combination as an object can the students conceive of many linear combinations of S at once. The student is then asked to conceive of not one, nor two nor three, but all linear combinations at once. They are asked to hold within their mind the infinitude of all such objects. That is span, built from the bottom up.

The second conception of span is a "top-down" construction. Here, one first conceives of a subspace. Unlike the above pathway, a subspace begins life as a formal object, defined in terms of properties. Furthermore, instantiations of this formal object are themselves typically infinite and there is no finite construction in the way there is for a linear combination. Once a student has internalized the formal object of subspace, the student can then begin to conceive of many subspaces at once. The student is then asked to conceive of not one, nor two, nor three, but all subspaces containing S at once. They are asked to hold within their mind the infinitude of all such objects and conceive of their intersection. That is span, built from the top down.

Both approaches, linear combination and subspace, are mathematically advantageous in appropriate contexts. For example, the former is useful for computation, while the latter is more tractable for formal proofs. The mathematician uses both. The student typically only confronts the first conception of span in their first course.

Concluding Remarks

This study examined a linear algebra instructor's textbook through the lens of ROGs, and, in particular, the instructor's beliefs and their knowledge of content and students. The instructor's textbook is a product, an accumulation of a sequence of numerous teaching decisions, motivated by their beliefs and in line with their goals to help students succeed in understanding linear algebra, to promote mental mathematical actions, namely, exploration, conjecturing, and proving.

When tasked with problem solving, the instructor had knowledge from his experiences as a mathematician of the importance of being able to sort through and retrieve relevant knowledge (in particular, definitions and propositions) in the context of proving. Furthermore, from his experiences teaching, as well as familiarity with mathematics education literature, the instructor anticipated that students would significantly struggle (to the detriment of their learning) with the spontaneous retrieval of relevant definitions and propositions. The instructor therefore made the teaching decision to include these references in the hints. Anticipating the complexity of certain full proofs, the instructor made the decision to recommend specific intermediate results in the hints. Based on their beliefs as a mathematician and their knowledge of mathematics education literature and teaching experiences, the instructor made the decision to make pedagogical content knowledge available to their students. We question, what impact engagement with mathematics education literature might have on student learning (metacognition)?

The instructor got the impression that the hints, in harmony with their other teaching actions, had some long-term impact upon (1) students' understandings of linear algebraic concepts and (2) how students approach proofs. Certain common definitions and propositions, such as the definition of subspace (Definition 4.24 in Figure 2) were invoked often throughout the term, and the students seemed to internalize them and speak of them fluently. After the course, students told the instructor that they learned that a well-written proof should include proper invocation of definitions and propositions.

Currently we are in the process of analyzing students' data, for example, the course journals, and exit surveys. The team intends to look at the structure of students' proofs, and how they constructed their proofs from the provided hints/resources. Without a doubt, proof is an important component of an abstract second course in linear algebra and more research on instructors and students is needed.

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