A Preliminary Investigation of the Language of Proof in Mathematical Textbooks

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Mathematical proof writing is known to follow various norms and conventions set by the mathematical community. Yet, recent scholars have questioned if these norms and conventions hinder the progression of the field to be inclusive in proof courses. In this report, we discuss preliminary findings of a linguistic analysis on the degree of gendered language used in eight undergraduate textbook proofs ranging in content. We report a mixture of gendered language use in mathematical proof writing.

Keywords: proof, language, gendered communication

It is largely accepted that the language of mathematical proof is genre-specific and follows particular norms and conventions (e.g., Burton & Morgan, 2000; Dawkins & Weber, 2017; Lew & Mejia-Ramos, 2020). A small body of research has attempted to empirically investigate these norms such as Burton and Morgan's work exploring mathematical texts and interviewing mathematicians related to elements of proof related to authority or Lew and Mejia-Ramos's investigation of a set of conventional breaches and mathematicians' evaluations of them. Such work points to the fact that there are observable linguistic features of mathematical proofs that can be studied. Further, these features do not only involve the structure and nature of argumentation; they primarily concern language in which proof is conveyed.

Recently, Weber and Melhuish (2022) argued that the adherence to language conventions may be in tension with critical and inclusive aims in mathematical proof courses. Indeed, scholars such as Brown (2018) have illustrated the ways that students who are first generation and urban may use more "varied and at times colorful dialects" (p. 7), losing advanced mathematical contributions because they are not communicated in a standard way. It is possible that the conventions of proof writing are not due to epistemic aims entirely and might systematically exclude certain individuals. To make this point, Keith (1998) argued that proof characteristics (such as being impersonal and directive) may align with typical ways men communicate rather than women. However, to our knowledge, there has been little empirical work on how proof texts may be masculine or genderized. With this background in mind, the aim of this preliminary report is to build on the work identifying language features of mathematical proof and to begin an exploration related to Keith's (1998) observation. We focus on two research questions: (1) What are the language features of proof found in undergraduate textbooks? (2) To what degree do these features reflect types of communication associated with gendered communication?

The Norms and Language of Proof

A number of scholars have pointed to the language and norms of proofs being distinct from other types of language. Selden and Selden (2014) refer to the distinct stylistic and linguistic features as part of the genre of proof. Recently, Dawkins and Weber (2017) reflected on the

values and norms of proofs both held by professional mathematicians and in mathematics classrooms. Of particular relevance to our examination of proofs, they identified, "Mathematical knowledge is justified by a priori arguments" and "Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author." Upholding these values is then reflected in linguistic choices in proofs. For example, they identified that proofs are written in such a way to reference the author or reader's agency. So, a proof would not contain an "I" statement, but rather impersonal "we." Such choices reflect the value of a priori arguments and that mathematical proof is meant to be universal and dependent on a particular author.

Perhaps in contrast, other scholars, such as Burton and Morgan (2000) argue that, "Conventions of mathematical writing are neither necessary nor natural consequences of the nature of the subject matter" (p. 450). In fact, in Burton and Morgan's analysis of published proofs and mathematician interviews, they identified similar conventions as Dawkins and Weber (2017), but also noted that authority can be found in proofs. For example, "terms such as clearly and obvious are relative to the individuals using them" and Burton and Morgan further suggest, "the extent or absence of such words is one of the interpersonal aspects of the writing that will influence the ways in which the readers of the text will construct an image of the author and will consequently judge the worth of the text itself."

The fact that there are some conventions that are not necessary to the subject matter is unlikely to be contested. For example, Lew and Mejía-Ramos's (2019) norm breaching study identified a number of breaches that mathematicians found unacceptable, especially in the context of textbooks. These include, "using non-statements, overusing variable names, lacking punctuation and capitalization, mixing mathematical notation and text, lacking verbal connectives, using formal propositional language, using unclear referents, and using an unspecified variable with an existential quantifier" (p. 55). While some of these breaches may hamper interpretability or validity of a proof, others are merely stylistic.

The imposition of linguistic and stylistic features that are not necessary to the validity of mathematical arguments has led several researchers to wonder about this impact for students (e.g., Brown, 2018; Tanswell & Rittberg, 2020; Weber & Melhuish, 2022). In Tanswell and Rittberg's reflection, they explain the concern with the particularities of mathematical language in proof (e.g., the interweaving of symbols and words, avoiding the first-person "I", the unusual "Let n be a number" style constructions, specialized use of imperatives and instructions, and being devoid of many features of natural language):

> This has the effect of elevating the regimented mathematical language that has grown up over the centuries to the level of being the sole vessel of objective mathematics. ... Once we realise that learners do not come from uniform linguistic backgrounds it quickly becomes apparent that the allegedly impartial language of mathematics is easier to access for some students than others (p.1202).

Brown (2018) illustrated this point sharing data from an inquiry class where majority of students were first generation and urban. She pointed to the "colorful dialects" they used and the ways they stand in contrast to standard ways of communicating math in proofs. Students' rich mathematical arguments may not been seen as important contributions if they are not structured in a way to align with conventional preferences. Overall, the literature base points to the fact that there are strong conventions in how proofs are written and that this is an important element to study because they have the potential to disenfranchise some students more than others.

Theoretical Framing

We take on the perspective that language reflects culture, and more broadly that language can be treated as a cultural artifact itself (Maltz & Borker, 1982). The culture in which one develops their language shapes their communication choices and how they interpret others. As noted by Maltz and Borker, these communication differences are salient across different ethnic backgrounds. Furthermore, the culture of language is not just developed at a global level (such as American English), but within gender, class, and other subgroups. For the scope of this paper, we provide particular attention to gender, and adopt Mulac et al.'s (2001) view of gender-as-culture. They suggest that, "much of what people know about interpersonal communication is learned from same-sex peers during the ages of 5 to 15 years." This then leads to communication differences across gender-lines. We note that this sociological interpretation does not mean that power dynamics are not at play for how and why there are communication differences, rather that it provides a mechanism for their existence.

This allows for identifying distinct language and communication aspects that may reflect different subcultures. For example, women may be associated with communicating in ways that involve more uncertainty and hedging while men may be associated in communicating in more direct ways. We further make the assumption that the language of proof reflects a distinct culture that may or may not align with communication elements salient in how different genders communication. While there are some that conjecture proof is more masculine in communication features, this remains an empirically open question.

Methods

In order to begin to analyze language features of proof, we began with the framework developed by Mulac et al. (2001). This framework stemmed from the synthesis of a number of studies identifying differences in how men/boys and women/girls communicate in both written and orally. The features can be found in Table 1. We note they align with four verbal communication style dimensions: direct versus indirect, elaborate versus succinct, personal versus contextual, and instrumental versus affective (Gudykunst & Ting-Toomey, 1998). The direct-indirect style captures the degree to which one explicates their intention through verbal communication. Elaborate-succinct style describes the richness and expressiveness of the language used, ranging from the most expressive (i.e., elaborate) to the most understated (i.e., succinct). Along the dimension of personal-contextual, a personal verbal style heavily rests on the use of personal pronouns (e.g., we, us) and locatives (e.g., then, there) whereas a contextual style deemphasizes the use of these language features. Instrumental communication style is goal-oriented and relies on the speaker (sender-oriented) to make the message clear. Alternatively, an affective communication style is process-oriented and the responsibility falls on the listener (receiver-oriented) to read contextual cues to decipher the message.

Table 1. Features of Language Found in Empirical Studies (Mulac et al., 2001) where M: male features, F: female features; X: mixed results from empirical studies.

Language Variable	Definition
Elliptical sentences (M)	A unit beginning with a capital letter and ending with a period in which either the subject or predicate is understood
Questions (X)	But not including directives in question form
Tag questions (F)	An assertion that is followed by a question asking for support
Directives (M)	Telling another person what to do

Negations (F)	A statement of what something is not
Sentence initial adverbials (F)	Answers the questions: how, when, or where? regarding the main clause [why and how long]
Dependent clauses (F)	A clause that serves to specify or qualify the words that convey primary meaning
Oppositions (F)	Retracting a statement and posing one with an opposite meaning
Judgmental adjectives (M)	Indicating personal evaluations rather than merely description
Uncertainty verbs (F)	Verb phrases indicating lack of certainty or assuredness
Progressive verbs (X)	Verbs presented in the "-ing" form
Intensive adverbs (F)	Adverbs such as "really" or "so"
Hedges (F)	Modifiers that indicate lack of confidence in, or diminished assuredness of, the statement
Justifiers (X)	A reason is given for a previous statement
References to emotion (F)	Any mention of an emotion or feeling
References to quantity (M)	References to an amount or quantity
Locatives (M)	Usually indicating the location or position of objects
Personal pronouns (X)	Words that stand for beings
"I" references (M)	First person singular pronoun in the subjective case
Fillers (X)	Words or phrases used without apparent semantic intent

We selected four common undergraduate textbooks across different subject areas: Smith et al.'s (2004) *A Transition to Advanced Mathematics*, Gallian's (2012) *Contemporary Abstract Algebra*, Fitzpatrick's (2009) *Advanced Calculus*, and Munkres' (2000) *Topology*. We then randomly selected a number between 1 and 10 (4), and selected the fourth proof in each text and the fourth proof after the halfway point in each text. Each of the four members of the research team applied the framework from Table 1. We met to reach consensus on the meanings of codes, discuss borderline cases, and ultimately arrive at finalized coded versions of the textbook proofs.

Results

We begin by providing some of the expected results in terms of features of proof language. None of the proofs contained: *elliptical sentences* (M), *questions* (F), *tag questions* (F), *uncertainty verbs* (F), *reference to emotion* (F), *"I" references* (M), *fillers* (X), *intensive adverbs* (F), or *hedges* (F). We note that this evidence points to a particular way of communicating in proof that does not contain many features of communication in other contexts. We also note that proofs do not contain some common masculine ways of speaking (elliptical sentences and I statements) nor contain many feminine ways of speaking such as reflecting questions and uncertainty or references to emotions.

We now consider the language codes that did appear in proofs beginning with the most universally found. All proofs had initial adverbials (F). These are sentences beginning with words or phrases serving the function of adverbs. "Thus" or "Therefore" are initial adverbials, as are clauses such as, "By Theorem 16.1," or "If I = {0}." On average, each proof had 6 initial adverbials (ranging from 4 to 10). Thus, we suggest that the use of initial adverbials is standard feature of proof and seems to serve the purpose of chaining one logical claim to the next. We contrast these with *justifiers* (X). Justifiers, as defined in the literature, provide a reason for

something *after* it is stated. Only one of the eight proofs had any justifiers with the first of the topology statements having two sentences containing "for" statements. The first was, "The case which B=∅ is trivial, for there cannot exist a bijection of the empty set B with the nonempty set {1, ...,n}." Otherwise, any explicit warrants were found before the inference being warranted. That is, they were found in initial adverbials with language such as, "Since X is normal,..." or "Because the theorem..." So, while justification is common in proofs, it is not provided as a justifier after a statement, but rather anticipates the statement that will be justified.

Other features found across all proofs were *dependent clauses* (F) (average of 3) and the *personal pronoun* (X) "we." Dependent clauses are more frequently found in the communication of women than men, but it is unclear whether the average of 3 is more or less than a baseline. The existence of "we" across all the proofs is hardly surprising as the literature has pointed to the commonality of the royal or impersonal we. We do know that "we" and one instance of "us" were the only personal pronouns in the proofs. We found no uses of the pronoun "I".

We also found that all, but one proof (7 of 8) contained *directives* (M). These directives begin with, "Let," "Suppose," "Assume," "Show," "Define," "Note," and "Choose." We note 23 such instances indicating that the commonality of directives mentioned in literature seems to bear out our analysis of the undergraduate proofs.

We quickly note the remaining features' prevalence. *Quantity* (M) was found in all eight proofs (usually references counts for abstract quantities such as "infinite"), but this seems like a contextual feature rather than a convention feature. *Location* (M) was not found in the proofs (although reference set membership was common). Again, we hesitate to say this is anything but a context feature. We also did not observe any *oppositions* (F) occurring within sentences as described in the framework, but we note there existence as complete new sentences in proofs by contradiction. We found that half of the proofs we analyzed had *negations* (F), that is defining something by what it is not such as "b-1/2 is not an upper bound." Three of the eight proofs contained progressive verbs (e.g., "consisting", "covering") reflecting a use, but not frequent use of these. Finally, we note the use of *judgmental adjectives.* One proof contained "trivial" while another contained "it is clear" reflecting at least the existence of these authority-driven language even within undergraduate texts.

Discussion

Our preliminary analysis explores the extent that language features from the linguistic literature are present in proofs. In the eight proofs we explored, we found that the impersonal "we" and directives were common in these proofs. We did not find any instances of uncertainty, including hedges, qualifiers, and leaving questions, nor did we find any references to emotion, oppositions, or the personal pronoun "I". While justification was common, *justifiers* (i.e., phrases used to justify prior statements) were rare. It is interesting that some features of proof align with masculine ways of communicating, such as the giving of directives, the lack of references to emotions, and the lack of hedges and oppositions. However, other features, such as the absence of "I" statements, are associated with the literature as feminine ways of speaking. At this point, it would be premature to make any claims about the degree to which mathematical proofs reflect masculine ways of speaking. Future research may involve a corpus linguistics analysis to document more accurate prevalence of these features, a convention breach analysis to see which of these features are considered salient to mathematicians, and comparisons to other texts both academic and non-academic. This is especially important when documenting prevalence of features such as dependent clauses. Dependent clauses are found more often in communication

by women, but that is a comparison and not a claim that men do not use dependent clauses. It would be useful to consider how these features in proof compare to some baseline texts that are not proofs.

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