Multivariational Reasoning in Linear Algebra:
How Mathematicians Reason about Linear Transformations

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Student difficulties in introductory linear algebra courses are often attributed to the novelty of the concepts in the course and the disconnectedness of these concepts to students' prior mathematical experiences. However, researchers have stated that a strong prerequisite understanding of the function concept is essential to students' understanding of linear transformations in linear algebra. In calculus and related courses, covariational and multivariational reasoning has been determined to be necessary for students to appropriately reason about functions in two or more variables. However, research on multivariational reasoning in linear algebra, especially with respect to linear transformations, is scarce. To contribute to the literature, we designed a study exploring a hypothetical model of how students might come to reason multivariationally about linear transformations. In this preliminary report, I discuss a part of the study – interviews with seven mathematicians who have either taught linear algebra or used linear algebra in their research.

Keywords: Linear Algebra, Linear Transformations, Multivariational Reasoning

Linear algebra is an important topic for many STEM students, including those studying computer science, data science, engineering, mathematics, and physics (Stewart et al., 2022). Recently, the second Linear Algebra Curriculum Study Group (LACSG 2.0) published recommendations for linear algebra instruction, taking into account advances in the field, industry, and technology over the last 30 years. Among the recommendations was the removal of calculus as a prerequisite for introductory linear algebra courses (Stewart et al., 2022).

In response to recommendations put forth by the first Linear Algebra Curriculum Study Group in 1993, Dubinsky (1997) claimed that students' struggles with mathematical concepts that are not necessarily part of linear algebra curricula but that are foundational to understanding linear algebra topics are one of the primary sources of students' difficulties in the course. In particular, Dubinsky noted that a strong prerequisite understanding of the function concept is essential to students' understandings of linear transformations. Additionally, Dorier et al. (2000) attributed students' conceptual difficulties in linear algebra to their difficulties with the "avalanche of new words, new symbols, new definitions, and new theorems" (p. 95) and the lack of connection to what students have already encountered in their mathematical experiences. Considering these views, it is important to connect students' learning of the linear transformation concept to students' prior experiences with the function concept, especially if linear algebra precedes calculus in the college curriculum.

There exist studies exploring the connection between students' understandings of the function concept and their learning and understanding of linear transformations (Andrews-Larson et al., 2017; Bagley et al., 2015; Zandieh et al., 2012; Zandieh et al., 2017). Much of this work has focused on a comparison of the concept images that students come to possess for functions and linear transformations. As part of a larger study, Zandieh et al. (2017) characterized student responses into three categories of mathematical structures (computations, properties, and clusters of metaphorical expressions) to compare their concept images of these concepts. In their work, the researchers found that "a reliance on properties appeared to impede students' development of a unified concept image of function while an ability to draw on metaphors facilitated such a development" (p. 36). That is, when students leveraged metaphorical linguistic expressions in their reasonings they were more successful in understanding various functions and transformations as examples of the same phenomenon.

In calculus and related coursework, covariational reasoning has been determined to be a vital way of thinking for understanding of functions (e.g., Castillo-Garsow, 2012; Oehrtman et al., 2008; Tallman et al., 2021; Thompson & Carlson, 2017; Thompson & Harel, 2021). Covariational reasoning encompasses the mental actions involved in reasoning about how two related quantities change in tandem with respect to perceptual time. When able to engage at the highest levels of covariational reasoning, a student can envision changes in one quantity's value as happening simultaneously with changes in another quantity's value and envision both quantities varying smoothly and continuously. Being able to coordinate the dynamic relationship between two quantities as they covary is crucial for students to be able to reason about concepts relating to functions, including derivative and accumulation (Silverman, 2017).

In linear algebra, Turgut (2019) utilized a dynamic geometry environment (DGE) to promote student understanding of matrix representations of geometric transformations as representing co-variation of inputs and outputs, consistent with a covariational view of functions. A review of the literature reveals that this study is the only one to-date that focuses on conceptualizing co-variation of two mathematical objects in linear algebra. However, the covariational reasoning framework was not utilized as a lens through which the author analyzed the data. That is, the study was not concerned with *how* or *to what extent* the students reasoned covariationally. Rather, Turgut (2019) employed the use of a semiotic mediation lens to investigate students' development of a conceptualization of matrices as representing geometric transformations, where treating a function as co-variation of independent and dependent variables was the byproduct of the use of the DGE.

More recently, there has been a greater focus on students' reasoning about the simultaneous change in more than two variables, excluding conceptual time as mediator, in calculus and other STEM subjects; this reasoning has aptly been referred to as *multivariational reasoning*. The development and evolution of multivariational reasoning as a framework of mental actions is rooted in the conceptual analyses reported on by Jones (2018). This theoretical report was "meant to form the basis of future empirical work" (Jones, 2018) centered around multivariational reasoning. Jones's first conceptual analysis was focused on analyzing a large set of functions and formulas derived from a collection of STEM textbooks. In particular, Jones considered how the variables in these functions and formulas "could be conceptualized as changing with respect to one another" (p. 1111) simultaneously and interdependently (Castillo-Garsow, 2012; Jones, 2018). The analysis included an identification of four potential types of multivariation inherent to mathematical functions and formulas: *independent multivariation*; *dependent multivariation*; *nested multivariation*; and *vector multivariation*.

To date, no studies have employed use of the multivariational reasoning frameworks to linear algebra concepts outside of vector field tasks presented in the context of differential equations (Jones & Kuster, 2021; Kuster & Jones, 2019). However, there are implications for the teaching and learning of linear algebra concepts when considering vector multivariational reasoning, even when viewed as the composition of parallel independent multivariations. Jones and Jeppson (2020), in particular, noted many similarities in the various types of reasoning within multivariation contexts. And "that it might not be necessary for students to learn about each type of multivariation separate from the others. By learning to reason about one type, they may simultaneously be developing reasoning abilities that transfers to other types" (p. 1146).

If it is indeed the case that the mental actions involved in multivariational reasoning are transferrable across the different types, then it may be that multivariational reasoning could serve as the catalyst for connecting students mathematical reasoning across courses and disciplines. In particular, the teaching of linear algebra concepts, such as linear transformations, from a multivariational reasoning lens might provide additional support and opportunities to make connections to prior mathematical experiences and concepts, such as reasoning covariationally about the function concept. In this study, I aim to explore this idea by addressing the research question: *What would be a hypothetical model of how students might come to reason multivariationally about linear transformations?*

Theoretical Underpinnings

Quantitative reasoning, "conceptualizing a situation in terms of quantities and relationships among quantities" (Carlson et al., 2002, p. 425), is an important component of students' mathematical learning. In this framework, a student has conceived of a quantity when they have conceived of a measurable attribute of some object; in this way, the quantity is idiosyncratic to the student conceiving of it (Thompson et al., 2017). When a student mentally unites the attributes of two or more quantities to make a new conceptual object that simultaneously represents both original quantities, this new object is referred to as a *multiplicative object*. When the variations of two quantities, coordinated to form a multiplicative object, are conceptualized in tandem, we call this *covariational reasoning*.

The idiosyncratic nature of quantity under these framings has allowed for the identification of mental actions pertaining to increasingly sophisticated levels of covariational reasoning. For example, the first identifiable mental action in the covariational reasoning framework is "coordinating the value of one variable with changes in another" (Carlson et al., 2002). Similarly, researchers have begun identifying the mental actions pertaining to increasingly sophisticated levels of multivariational reasoning (e.g., Jones, 2018). In working to identify potential types of multivariation and associated mental actions, Jones (2018) conducted two conceptual analyses.

Conceptual analysis is a tool that leverages the tenet of radical constructivism that knowledge persists because it has proven viable in the knower's experiences. Because, as mathematics education researchers, we seek to improve the learning attained by all who study mathematics, the purpose of a conceptual analysis is to develop a model of how a concept may be structured, ways of knowing the concept that may be favorable to learners, or to develop models of what learners actually know at a moment in time and understand in specific situations (Glasersfeld, 1995; Thompson, 2008). In discussing the role of conceptual analyses in developing hypothetical models of a concept and how students might come to reason about it, von Glasersfeld (1995) stated:

> [I]t is indispensable to have a fairly explicit model of what these concepts might be in the adult. Mathematics textbooks are not very illuminating in that regard and philosophers of mathematics rarely stoop to say anything about the conceptual raw material of their construction. (p. 161)

Accordingly, I designed a study to investigate a hypothetical model of how students might come to reason multivariationally about linear transformations by first performing an initial conceptual analysis leveraging several undergraduate linear algebra textbooks. This analysis guided my work on designing interview protocols to elicit multivariational mental actions of mathematicians.

Methods

In this report, I share a part of a larger study in which I investigate mathematicians and students' multivariational reasoning in linear algebra, with an aim of developing a hypothetical model for how students might come to reason about linear transformations. To achieve this goal, I first conducted interviews with mathematicians. The purpose of the interviews with mathematicians was to identify the mental actions that they call upon when reasoning multivariationally about linear transformations. Then, I plan to refine the interview protocols to conduct interviews with students to again identify the mental actions that students call upon when reasoning multivariationally about linear transformations. With the results of from these data sets, I aim to develop a hypothetical model for how students might come to reason about linear transformations.

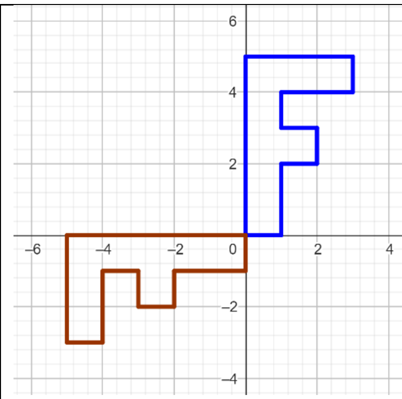
At this stage in the project, seven mathematicians each participated in two 60-minute, one-on-one, semi-structured interviews centered around the concept of linear transformation. Mathematicians were considered suitable for participation in the study provided they had previously taught a course in linear algebra, or their research had leveraged linear algebra knowledge.

The first interview was used to familiarize the researcher with each mathematicians' background in linear algebra and determine how the mathematician participants reasoned multivariationally about linear transformations represented algebraically. These algebraically represented transformations were determined to be more similar in appearance to functions that a student might have prior exposure to before taking an introductory linear algebra course. For example, after being asked which given transformations could be linear, mathematicians were each asked to describe the relationship between x and $T(x)$ for the transformation $T(x) = (x, -3x)$. If there was no discussion of varying quantities, then a follow-up question at this point was "What would happen to $T(x)$ as you changed/varied the value of x?". This prompting was used to elicit multivariational reasoning, so that the mental actions of multivariational reasoning that mathematicians engaged in could be identified during data analysis and used to inform the hypothetical model for students' multivariational reasoning.

In the subsequent second interview for each mathematician, participants were similarly asked to reason about matrix transformations and geometric transformations; an example of each is shown in Figure 1. For each type of representation, mathematicians were first asked which given transformations could represent linear transformations and were then asked to reason about the relationships between inputs and outputs for particular transformations. During task design, it was hypothesized that reasoning about the images of the standard basis vectors and the domain and range of a linear transformation might help students reason more powerfully about the relationship between the inputs and outputs of the transformation. This led to development of sub-questions ii. and iii. for the matrix transformation tasks (Figure 1). These sub-questions were given iteratively to the mathematicians, provided the ideas had not been brought up authentically by the participant while engaging in previous parts of the task. Throughout both interviews, each mathematician was frequently asked to reflect on anticipated student difficulties with each task.

Given the matrix $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$, determine whether the transformation given by $T(x) = Ax$ could be a linear transformation.

i. What happens to the output vector as you vary the input vector?
ii. What does A do to a vector?
    a. What does A do to the basis vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? To $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?
iii. What is the domain and range of the transformation?



What effect does the transformation have on an arbitrary vector?

Figure 1. An example of a matrix transformation task (left) and a geometric transformation task (right) given to mathematician participants during their second interviews.

During each interview, an over-the-shoulder camera was used to collect data relating to mathematicians' physical cues (e.g., hand gestures, motions, and written work). Mathematicians' written works were captured in real time using screen recording on the interviewer's iPad Pro®; data from the camera and screen recording for each interview was then superimposed for the proposed of data analysis.

In the initial coding of the data, I am identifying and categorizing mathematicians' reasonings (e.g., reasoned geometrically) and potential related mental actions (e.g., recognized one quantity's value as being dependent on another) from the overlayed video and transcribed audio. Next, I will identify properties of these categories (e.g., whether a mental action was task or representation specific). Following this initial analysis and coding of mathematicians' reasonings, I will perform a member check with each participant to ensure that their views and reasonings are accurately reflected. During axial coding, subcategories of mental actions will be related to categories of mental actions to which they are subordinate to or supporting. I will also examine subcategories of mathematicians' mental actions to determine whether any categories are repetitive and can be removed, condensed, or combined.

Mathematicians' responses to anticipated student difficulties for each task will also be coded thematically and compared to existing literature on students' difficulties. Insight from mathematician responses and results from data analysis will be used to modify interview tasks and protocols ahead of student interviews.

Discussion

I will present data analysis results from the interviews at the conference. In preliminary initial coding, I observed that some mathematicians focused on varying the input of a transformation component-wise by small arbitrary changes to determine the effect this had on the corresponding output vector. It seems that their reasonings included recognizing in/dependence and coordination mental actions. The audience will be asked to discuss some interview questions to elicit additional mental actions that I may have not observed in my data set.

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