Student Use of Series Expansions as an Approximation Technique in Physics Modeling Tasks

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As part of a larger project to investigate student use of mathematics in upper-division physics courses, we have examined how students use series expansions as a means of approximating and simplifying complicated expressions in theory-oriented physics courses. Student responses were collected on pre- and post-instruction written tasks in which students were prompted to use a series to approximate an expression arising from a problem in electricity and magnetism. Despite prior experience with series in calculus and physics, students struggled to determine appropriate quantities in which to expand and did not attend to units or the convergence of their series. While student success rates improved after targeted instruction, many students expanded in quantities that were neither dimensionless nor small, and thus unproductive for modeling.

Keywords: calculus, series, physics, applications

This work is part of a project to investigate student use of mathematics in upper-division physics courses. A key lens has been the study of how concepts studied in calculus and other mathematics courses are implemented in modeling tasks in physics, and how students perceive and take up these concepts given the different disciplinary context. For this study, the research questions are: To what extent are upper-division physics students successful in using series to model simple physical systems? What aspects of this task are challenging? What implications does this have for instruction, both in physics and in mathematics?

Background and Previous Research

Power series are commonly covered in an introductory calculus sequence. In physics, Taylor and Maclaurin series are used frequently as modeling tools, for simplifying analytical solutions and constructing approximations. As part of a project to study student use of mathematics in upper-division physics and support curricular interventions, we have investigated the reasoning used by students in using series to manipulate physics expressions and use mathematical tools for modeling physical systems, including the use of series as approximation tools.

While often physics students will be asked to perform a full Taylor series expansion, in many cases it will suffice to use a binomial expansion. A typical usage is to generate an exact expression from a physical and mathematical model, then to replace a portion of this expression with a power series, in powers of a small, dimensionless ratio of two physical quantities. For example, for the expression for the electric field of an electric dipole (a pair of charges with equal absolute value but opposite sign), shown in Figure 1, the task directs students to consider points far from the dipole. The statement 'far from the dipole' can be interpreted to mean that the distance y is considerably greater than the distance d between the two charges, and thus mathematized as $y>>d$. Students can perform algebraic manipulation to produce an expression proportional to $(1-d/2y)^{-2}$, which is of the form $(1+x)^p$ with $|x|<1$ as stated in the course text (Boas, 2006). With $|x|<1$ the series will converge, and higher-order terms fall off quickly, allowing the complicated expression to be replaced with a relatively simple polynomial.

Previous research on student learning of series comes from both the mathematics education and physics education research (PER) communities. RUME studies have tended to focus on conceptual understanding and visualization. Alcock and Simpson (2004) focused on use of visualization of series. Martin and Oehrtman (2011) investigated the concept of convergence in terms of metaphors, including a part / whole metaphor and a cutting metaphor. Champney and Kuo (2012) focused on visual images of the Taylor series as an approximation.

PER studies have often focused instead on application to problems, Smith et al. (2013) reported that students lacked fluency in both creating and using series, despite prior exposure in mathematics and physics as well as evidence of conceptual understanding of the graphical meaning of series terms. Wilcox et al. (2013) investigated Taylor series as a modeling tool in upper-division classical mechanics. From this analysis they developed the ACER framework for student use of mathematics in physics, named for its four components: activation of a tool, construction of a model, execution of the relevant mathematics, and reflection on the result. Student difficulties in each of the four phases were described, including those with expansions about a point other than zero and a lack of meaningful reflection. In both PER studies, students had to make choices about the series expansion, but in neither case was it necessary to manipulate an expression to produce a small, dimensionless ratio with which to expand.

Context and Methods

Context for the Research

This work has taken place in the context of a course on mathematical methods for physics ("Math Methods"), a required course in many physics departments that is generally intended to prepare students for the mathematics encountered in upper-division theory courses. Course prerequisites include three semesters of calculus and two semesters of introductory physics; all students would have encountered series and the relevant physics concepts. For this report, we describe data that were collected in twelve in-person sections of the course from different semesters over the period 2009 to 2022, all taught by the same instructor at a comprehensive university serving a diverse student population. Instruction focused on mathematical ideas used as modeling tools in physics and included relevant post-tests on course quizzes. Total enrollment over these semesters is approximately 160 students and the performance was roughly similar across sections while accounting for fluctuations. Approximate demographics for these semesters were: 78% male, 22% female, 45% white, 33% Latino, 18% Asian.

Instruction in Math Methods was interactive and used instructional materials developed in response to both prior research and results from early versions of the ungraded quiz. Students in small groups worked on guided worksheets. Tasks in the worksheets included qualitative questions about the first three terms in a Taylor series for several points on an arbitrary graph (adapted from Smith et al., 2013), followed by procedural tasks including evaluation of terms of series for sine and cosine. Students were then asked to interpret their results by comparing their series for sine and cosine to the original functions in terms of values and behavior (slope, curvature). Student groups then revisited the dipole task and executed the expansion, with guiding questions that directed their attention to the units and magnitude of the expansion variable and what implications this had for series convergence and meaning. The purpose of this report is to examine aspects of student reasoning, but the results also have implications for the instructional materials that are addressed in the discussion section.

Method

For this study, we present the results of written responses to tasks administered as part of course assessments, including ungraded and graded quizzes. Each task involved the use of series to derive an approximation from an exact analytical result. In all cases, the generic form of the

Consider the electric field at points along the axis of an electric dipole, $E = \frac{+kQ}{\left(y-\frac{d}{2}\right)^2} + \frac{-kQ}{\left(y+\frac{d}{2}\right)^2}$ for a point far from the dipole, with y much larger than d.

The first term of the expression is $\frac{+kQ}{\left(y-\frac{d}{2}\right)^2}$. Use the binomial expansion (given) to construct the first three terms of a series for this term given that $y >> d$. Explicitly identify what x and p you are using. What simplification is allowed by the fact that $y >> d$?

Figure 1. Task from initial ungraded quiz. A diagram was included (not shown due to space considerations). The problem statement in the quiz included the expression for the binomial expansion from the course text (Boas 2006), with the form $(1+x)^p$. *Graded quizzes were similar but included different contexts (E field from charged ring, potential from charged disk).*

terms of a binomial series was available to students, either as explicit information included in the problem statement, or as a supplemental sheet of equations (e.g., in given information on a quiz).

The initial task for the ungraded quiz involved a series expansion for an expression for the electric field of a dipole for points along the axis containing the two charges (Fig. 1). This task is often encountered in introductory electricity and magnetism as an example and/or textbook problem (Halliday et al., 2011), and the approximate field proportional to the inverse cube of distance is familiar to experienced physicists. Graded quizzes after instruction included other tasks of similar structure (expression given, perform expansion to generate approximation) that are not shown due to space limitations. These tasks included four examples from electricity and magnetism, involving expressions for the electric field or potential for an extended object with a distributed charge (e.g., a ring or disk) or for a simple model of a crystal lattice.

Written responses were collected on an ungraded quiz (dipole, Figure 1, $N = 156$) on the first day of instruction in the Math Methods course over twelve semesters. Additional responses were collected on the first graded quiz (other tasks, not shown, $N = 161$). Student responses were coded by the lead researcher and student assistants using qualitative inductive analysis (Otero and Harlow, 2009); the initial data were coded without *a priori* categories, based on correctness and the overall approach taken. Initial codes were then refined, informed by an analysis of the steps in the task. Categories are described in results, below.

Results

Categories coded for included: mathematization of relative distances (e.g., $y>>d$), choice of expansion variable, dimensionality and magnitude of expansion variable, identification of exponent, execution of expansion, and interpretation of results (see Table I). For each category, specific choices in the response were recorded, e.g., whether to expand in powers of (y-d/2) or to factor the expression and expand in powers of $d/2y$ or $2y/d$. For example, expanding in (y-d/2) would be coded as neither small nor dimensionless. In contrast, $2y/d$ is coded as dimensionless but is not small compared to one. Nearly all codes of 'small' were also coded 'dimensionless.'

Responses on the initial ungraded quiz, before instruction in the Math Methods course, illustrated the difficulty of the tasks but provided relatively little insight into student reasoning. Very few students produced complete and correct responses to the task. A third of the responses

Table 1. Categories coded for in student responses, and examples of the coding. Examples are drawn from post-test questions involving a ring or disk.

Step	Examples
Mathematize	"far from the ring so z>>R"
Choose expansion variable	
dimensionless	"$x=R^2/z^2$"
small compared to one	"use z/L, <<1 b/c L>>z"
Identify exponent in binomial	"p=-3/2, negative brings to numerator"
Execute expansion	[omitted due to space constraints]
Interpret results physically	"we can cut out [terms] so →kQ/z which is a point charge from far away"

were either blank or nonproductive and uncodable, despite ample time to respond. Under 10% of responses transformed the expression into a form suitable for the expansion, i.e., $(1+x)^p$ with x dimensionless and $0<|x|<1$. Students who identified such a dimensionless quantity frequently neglected to attend to convergence, choosing a series with powers of a quantity large compared to one (e.g., y/d). Around 3% of students expanded in a dimensionless quantity small compared to one, one that would lead to a series that converges.

Student responses to the question about the significance of $y>>d$ further suggest the importance of the interplay between mathematical calculation and more qualitative physical reasoning. A large group of coded responses included answers stating incorrectly that the electric field would be zero. This category included two subcategories. One set of codes (~20% of students) included assertions, without calculation, that the electric field is zero because y is large. The second group of codes (~30%) included an algebraic calculation or symbolic argument, stating explicitly that $y>>d$ allows one to set $d=0$ or set $y-d/2=y$, in which case the expression for E reduces to zero. It is not clear whether the students in the second group force the mathematics to support their intuition that $E=0$ or simply perform the calculations and reach that result. The former might be explained with a dual-process model (Kryjevskaia et al., 2021).

Table 2 compares results before and after instruction. The post-test quizzes involved four tasks different from the dipole, all involving electricity and magnetism. The results are shown as a single combined entry because there was very little difference in response patterns. (For example, the fraction of students choosing a dimensionless x with which to expand was 72% in tasks involving electric potential of a disk, 78% in tasks involving electric field of a ring, and 76% in tasks involving electric forces in a crystal lattice.)

Table 2. Responses to series approximation tasks before and after tutorial instruction. In the table, x and 'exponent' refer to the binomial $(1+x)^p$ from the course textbook (Boas, 2006).

Instruction	Appropriate quantity x	x unitless	($\|x\|<1$)	Appropriate exponent
Pre (N = 156)	3%	8%	3%	3%
Post (N = 161)	56%	76%	60%	67%

More students successfully chose a dimensionless quantity with which to expand, but roughly 20% of those who did still chose quantities that were large compared to one, expanding in a series that would not converge and was not productive in modeling.

All post-test examples led to series in which there were fractional and/or negative exponents, which proved to be challenging. For example, given the expression for electric potential of a charged disk, $V = 2\pi k\sigma\left(\sqrt{z^2 + R^2} - \sqrt{z^2}\right)$, the students who answered incorrectly frequently expanded with a positive integer exponent (1 or 2, rather than $(1 + {R^2}/{z^2})^{1/2}$ with exponent ½).

While the algebraic manipulation required means that success on these tasks will be linked, there may also be a hierarchy of ideas; most of the students who correctly identified a small argument also chose a dimensionless argument and the correct exponent. The converse was not true; roughly one in three students who identified the correct exponent had incorrect series terms.

Discussion

Prior research may have underestimated the difficulty for students in using this technique by providing examples in which the variable of expansion was already dimensionless and small compared to one. Even after targeted instruction, nearly half of the students expanded these symbol-rich expressions in powers of a quantity that would not be productive for modeling.

While series are a point of shared emphasis in calculus and physics courses, the emphasis differs in important ways. Power series in physics must be dimensionless, but few students on the pretest chose to expand in a dimensionless quantity. Whereas a mathematics course might provide a 'cleaned up' expression to simplify student calculations (and instructor assessment), students in physics will encounter expressions that are symbolically rich and quantities that have units. Despite the strong emphasis in calculus on convergence, physics students need support to understand how this idea plays out in physics contexts. Students are likely to encounter an expression in a form inconvenient for expansion and must make choices consistent with convergent behavior that consider the dimensionality of quantities in physics. In our data set, many students have chosen to expand in powers of a quantity greater than one, for which a series would not converge and thus would not be useful in modeling. Instructors should be cognizant of the disciplinary differences their students will encounter and provide appropriate support.

The case of the dipole illustrates that it is sometimes necessary to perform an expansion rather than simply arguing that $y >> d$ implies that $d \sim 0$. While ignoring a small quantity might be productive in certain circumstances while modeling, here it eliminates the interesting physics of the dipole configuration, in which the small extra distance to one charge has important physical consequences, thus highlighting disciplinary practices.

We are using these results to modify the instructional materials and develop a set of modular curricular materials that can be adapted to the purposes of different upper-division physics courses. Preliminary materials seem to provide support for students in identifying dimensionless quantities but students seem to need additional scaffolding to recognize when and whether their series are convergent. Additional research and development in ongoing. Current versions of the materials are hosted at PhysPort.org.

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