Re-conceptualizing the Construct of Mathematical Autonomy: From Individual Trait to Quality of Action in Context

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In this paper, we motivate the need for a definition of mathematical autonomy that does not imply that autonomy is a trait of an individual. Rather, we demonstrate the usefulness of a lens on autonomy that captures the characteristics of actions taken in contexts that shape the space of sensible action. Shifting from trait to quality of action in context further allows us to analyze how actions of groups of individuals doing mathematical work can have qualities associated with mathematical autonomy. We ground our theoretical work in data from a longitudinal case study of an undergraduate student, Kaleb, whose orientation to mathematical sense-making changed sharply across multiple contexts of activity.

Keywords: Intellectual Autonomy, Mathematical Experience, Introduction to Proof, Upper-Division

Introduction

It has long been documented that U.S. students experience challenges of many sorts in moving from lower division courses to upper division courses (Moore, 1994). A challenge of particular interest is the movement from math coursework that is typically more algorithm and procedure oriented to upper division coursework that includes more focus on construction and validation of arguments and proofs (Selden, 2012). In light of the centrality of this challenge, a subset of the authors conducted a small-scale longitudinal study to investigate students' experiences as they navigated this challenge in the context of a large public U.S. university (Bae et al. 2018; Smith et al, 2017). This study focused on student experience and development of agency and autonomy through this transition to proof-intensive coursework in contrast to more traditional focus on mastery of proof methods and mathematical notation.

One way to capture an important facet of productive disciplinary engagement at the undergraduate level, and that is related in a key way to the transition from computationally-focused to proof-intensive work mentioned above, is that when students encounter a challenge in their mathematical work, they are able to consult perceived sources of mathematical authority (self, teachers, texts, internet resources) as appropriate and needed. This goal has been articulated in various ways in the literature discussing the need for students to be able to move away from appeals to external authority in determining validity of arguments (Harel & Sowder, 1998; Stylianides, 2007; Yackel & Cobb, 1996). Inglis & Mejia-Ramos (2009) caution that looking to authority figures, per se, is not a problem, but concur that students should be empowered to establish the validity of arguments through their own sense-making processes. That is, collegiate mathematics educators want students to come to see themselves as local disciplinary authorities that view the correctness of a proof as something they can assess via mathematical reasoning and mathematical logic, not something that they are dependent upon external sources of authority such as teachers or texts in determining.

In describing these goals for students and instruction, sometimes the term "autonomy" is invoked (NCTM, 2000) with the expectation that a goal for mathematics instruction is that students become increasingly autonomous as they move through their mathematics coursework and beyond. However, there are multiple perspectives on autonomy in the literature. Two prevalent perspectives relevant to students' educational experiences include psychological and intellectual autonomy. Psychological autonomy (Deci & Ryan, 1980) is linked with self-determination, free-will, and independence (self-reliance). Intellectual autonomy, with roots in the work of Piaget, casts autonomy as a trait which individuals develop over time, in which individuals become increasingly independent in sense-making in the context of the ideas of others (Piaget, 1973). Later, Wood reformulated the notion of intellectual autonomy through a communicational lens, building upon Sfard, to focus more upon the individual in relation to sources of authority and whether individuals see themselves as sources of authority, capable of deciding whether a given mathematical inference is right or wrong (Wood, 2016).

In this paper, we broaden the original focus on individuals making determinations of right or wrong inferences to students remaining personally involved in processes of sense-making (e.g., making determinations of whether actions taken in response to mathematical challenges make personal sense or not). We further expand from individual sense-making to sense-making in group contexts and also consciously expand from other people as trusted external authorities to include other kinds of external sources of authority such as internet-based resources, including AI systems.

Research Focus and Question

The data for our study was collected at a large public U.S. university where students, both mathematics majors and minors, first encounter mathematical proof formally in a semester-long, problem-based ITP course. In the ITP course, project data collection involves a baseline interview, a task-based interview, and a series of homework reflection logs from each of the 15 participants in the first cohort. Both the mathematics majors and minors in our study subsequently enroll in two or more advanced, proof-based mathematics courses to satisfy their major or minor requirements. Our longitudinal study involves following students into these subsequent courses as well as collecting data from a second cohort of students. Participants were selected to represent a broad spectrum of gender, ethnicity, major/minor, and mathematical beliefs as measured by the Mathematics Attitudes and Perceptions Survey (Code, Merchant, Maciejewski, Thomas & Lo, 2016). In addition to the interview and reflection log data mentioned above, roughly a third of the ITP classes were observed by members of the project team to provide context for the research team in interpreting the experiences reported by students in the interviews.

The specific question that guided this inquiry in this paper is "Which view of mathematical autonomy [trait or quality of action] is more useful in understanding collegiate students' progression through their upper-division mathematical coursework?"

Reconceptualizing Mathematical Autonomy: A New Definition

Our initial framing of mathematical autonomy followed Yackel and Cobb's (1996) analysis in seeing intellectual autonomy as both a *state of awareness of one's intellectual capabilities* and a *willingness to act on them in engaging in challenging mathematical work*. Where Yackel and

Cobb conceptualized intellectual autonomy solely within the social practices of mathematics classrooms, we needed our constructs to apply also to in–class and out–of–class mathematical activity, as college students have great freedom to act (or not) in ways they see as productive. Indeed, one key developmental task for college students is deciding how and why to deploy out–of–class time and activity productively.

Through our initial attempts to characterize the mathematical autonomy students were displaying at different points in time in their undergraduate experience, we became more attuned to the relationship between students and perceived sources of mathematical authority. That is, we came to appreciate how students' views of themselves as sources of mathematical authority was context-dependent, as well as students' views of themselves in relation to other sources of mathematical authority (instructors, other students, family members, or sources on the internet). In developing our definition, we previously analyzed students' shifting relationships to mathematical authorities such as family members (Castle et al., 2022) and students' usage and reflections on the usage of internet sources of authority (Levin et al., 2020). The working definition for mathematical autonomy that we settled upon is:

> *Mathematical Autonomy:* A <u>quality of action</u> that reflects the actor's active resistance to endorsing, following, or replicating the reasoning of mathematical authorities (e.g., texts, internet sources, instructors, peers) while engaging in a process of sense-making. The quality of action reflects the actor's judgment of using only the external resources necessary to continue a sense-making process when they meet a local obstacle and/or engaging in sufficient sense-making to transform the reasoning of others/other resources to their own.

Illustrating Features of Mathematical Autonomy: The Case of Kaleb

In this paper, we have chosen to illustrate the affordances of the new definition of mathematical autonomy using the data from a single case, Kaleb. Kaleb was a mathematics major, considering adding computer science as a second major or minor at the time of his Introduction to Proof course, when we began interviewing him. He had aspirations of graduate school in mathematics, and was possibly interested in teaching. By following Kaleb for multiple semesters, we found Kaleb displayed autonomy in different ways depending on mathematical context.

Kaleb's individual sense-making actions as autonomous in the ITP course

Early on in Kaleb's Introduction to Proof course, he showed more autonomous actions by actively engaging in sense-making and rejecting passively accepting others' mathematical authority. For example, in the ITP course, Kaleb thought the course was teaching them how students could think mathematically, and this aligned strongly with what he thought he needed to learn. Kaleb faced some challenges in doing homework, and his actions dealing with those challenges showed that Kaleb valued his work and tried to give him some time to think before he used other resources:

> Certain ones [tasks] took longer than others, but I found if I got stuck, I could get up, go get a drink, come back, play on my phone for a little bit, get my mind off of it. And then, looking at it with a fresh start, it would pop out.

He reported that if he had given enough thinking time for himself, he would reach out to other resources, such as peers or the course-provided online platform Piazza. In one instance, Kaleb reported in his homework log that he felt the instructor included more of a hint than what he thought necessary on Piazza. This led him to use only the first half of the hint and create his own outline for what he needed to do. After the whole process, he still spent another 40 minutes to an hour to write a proof. This excerpt shows Kaleb's autonomous action in this context in not accepting an authority figure's work when he was stuck, but rather trying to understand, evaluate, and use only the part he felt he needed to get unstuck. At this time, he expressed a strong view against copying solutions from other sources:

> If you are simply copying the answers down to get answers and submitting them as your own, then yes, that is cheating. However, looking for an answer and using it to get your own answer is fine… From what I can tell, one of the main themes about the class is teaching us how to think more mathematically. If you only copy someone, you let them do the thinking for you. You never learn anything and are wasting everyone's time.

Kaleb's joint work actions in ITP and Abstract Algebra as autonomous

When Kaleb could not solve problems, even though he gave himself some time to think about the problem, he consulted external resources. In the ITP course, the most used resource was a friend, Steven (pseudonym), whom he met in the ITP course and a statistics course. Kaleb and Steven regularly spoke on the phone and on Sunday night, and one explained to the other person as well as sometimes sending photos of their work when one of them was struggling.

Similarly, in Abstract Algebra, Kaleb worked with another friend who was also taking the same course in a different section but with the same instructor. Kaleb and the friend bounced off ideas with each other, and he described their contributions as 50-50. They sometimes had different ways of solving the same problem, and they examined each other's solutions and were content to submit their final work in different ways. We view this excerpt as showing that they did not just follow each other's work, but rather Kaleb's joint work with his friends in these contexts had features of mathematical autonomy.

Kaleb's use of internet resources as more and less autonomous, depending on context

In addition to human resources, Kaleb used internet resources, and his quality of actions shifted, depending on mathematical context. Above, we described how Kaleb resisted the reasoning of authorities when consulting internet-resources when stuck. However, Kaleb's actions towards internet resources changed in the following semesters. In the Linear Algebra course that Kaleb took concurrently with Abstract Algebra, he did not have a peer who could work together. He found himself consulting internet sources faster than in the ITP or the Abstract Algebra courses. He reported that once he spent 10-20 mins per question, he went to the internet to find some solutions. When he found a solution, he reported that 60% of the time he would try to make sense of what the found solution said and rewrite it in his own words. However, for the other 40% of cases, he reported that he copied the solution as he found it. Kaleb attributed the reasons for this action was that there were too many questions, each with little worth of his time because they were not a big part of his grade. In this course, homework was 8% of the final grade, and only some of the questions were graded in the course:

> I've definitely had it where I don't completely understand what's going on. […] It feels like cheating, or it feels…it doesn't feel good, but I know I'll get points and I'll get an okay grade.

Although there were external situations (i.e. having too many questions), it is important to note that in this context he started copying solutions from online sources. In the previous semester (in the ITP course), he viewed copying down others' solutions as relinquishing one's opportunity for sense-making. However, a less autonomous pattern of action emerged in Linear Algebra. His larger frame guiding action appeared to have shifted from understanding to earning points.

Later, in Spring 2020, when the COVID-19 pandemic erupted, Kaleb was taking Analysis I. Before COVID-19 changed the modality of the course, to do the homework in his analysis course, Kaleb said he spent 10 minutes thinking by himself and then left the problem to come back later. He said he usually repeated the cycle three times, then he tried to search online if he could still not solve the problem. Because the majority of the homework problems were from the textbook, Kaleb said it was very easy to find solutions on the internet. Also, he could find the solutions from other college courses or other professors. Even though he was using others' solutions, Kaleb reported that before COVID-19, he still valued understanding and sense-making because he knew that he would need them for the exams in the short term and for his career in the long term.

When COVID-19 happened, Kaleb had to go back to his home and had to go back to his parents' house, disrupting his sense of space and place (Küchle et al, 2023). His motivation for doing homework dropped significantly, which in turn influenced his actions, becoming less autonomous. Kaleb admitted that the quality of his actions before and after COVID-19 was different. Kaleb did not view his parents' house as a conducive workplace, and during this time he also suffered the loss of family. These changes made him focus on his personal life rather than schoolwork, which led him not to submit a few homework assignments, and to use the internet more on assignments he did complete.

Discussion

The data from the case of Kaleb illustrates how trait-based views of autonomy are insufficient for capturing the context-dependency of college students' actions in proof-based courses. As opposed to having a trait or characteristic of "being autonomous," Kaleb instead showed the actions he took could be cast as more or less autonomous depending on different contexts, such as his work with peers in his ITP and Abstract Algebra courses or the differences in his ways of using internet-based mathematics resources across his coursework. Further, as opposed to being a trait that increased in amount over time, we uncovered the way that circumstances and the constraints of his experience in different courses (e.g., different working conditions and course modality due to COVID-19) went hand in hand with him exhibiting less autonomous actions.

Thus, the data we present push the field to reconsider "autonomy" in a way that is not synonymous with a trait such as "individual independence." As opposed to framing autonomy as a trait that an individual may/may not possess, we are able to focus on qualities of actions taken during mathematical work. Our expanded way of framing autonomy focuses on qualities of action that align with being able to use resources (including social resources such as peers, and material resources such as the internet-based sources) appropriately. What is considered appropriate can align with current norms around collaboration and use of internet-based resources, both of which are features of modern work. Especially in the case of internet-based resources, the use of AI systems such as ChatGPT to assist in creative work, will only become a larger issue.

Methodologically, instead of assessing the amount of "autonomy" that an individual possesses, or even a Likert-based ranking, we can focus on records of specific interactions in specific contexts around mathematics. In conducting a case study of an individual over time, the focus shifts from differing amounts of "autonomy" across time to differences in the qualities of actions individuals take over time. One thing that complicates the tracing of qualities of action of individuals, is that the context in which actions are taken needs to be accounted for. The norms and practices in different course contexts can help one to explain how/why individuals take particular actions. For this reason, it is important for us to understand more about the constraints that participants perceive relative to their actions. We view this system of constraints as what constitutes the mathematical agency - the felt capacity for action - that students perceive (Levin, et al. 2020).

In terms of implications for instruction, our conceptualization of mathematical autonomy does not imply a restriction on the material and social resources available to students during their reasoning processes. In building the capacity to act with mathematical autonomy when faced with mathematical challenges, students need opportunities to engage deeply before abdicating their involvement in the resolution by, for example, calling over an instructor or other mathematical authority to direct them, or to search on the internet for the task they were working on and copying the argument. In undergraduate education, students' ability to navigate "stuck points" is linked with their ability to continue their engagement in mathematical challenges (Lu, 2021). Strategies for supporting productive struggle support the ability to act with mathematical autonomy (SanGiovanni et al, 2020; Warshauer, 2015).

Acknowledgments

This paper reports research supported by the National Science Foundation's IUSE program under grant #1835946. All opinions, findings, and conclusions or recommendations expressed here are those of the authors and do not necessarily reflect the views of the Foundation.

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