Exploring Graphical Reasoning from Revised Responses to Function Composition Tasks

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Students learn about function composition, $(g \circ f)(x)$, in secondary school. From two given equations, one might identify the composite function algebraically via substitution, $g(f(x))$. But what about when functions are given as graphs? This study aims to explore how students reasoned graphically with their revised responses to function composition tasks. We previously identified common types of resultant graphs participants generated in trying to sketch – in a short time period – the composition of various graphically-depicted functions. This paper specifically examines the ways in which students revised their graphs, when given more time to do so and after having engaged in trying to compose functions in a wider variety of contexts, and the reasoning that they provided for these changes. As such, we extend prior work by exploring how students reason about function composition when provided unlimited time constraints on their activity.

Keywords: function composition; graphical reasoning

Students are exposed to function composition in secondary school (cf., CCSSM, 2010). Primarily, they are taught that $(g \circ f)(x) = g(f(x))$ – meaning, for two functions defined by equations, one can procedurally determine the composite function via algebraic substitution. Yet, such algebraic approaches are limited in their ability to develop sufficiently deep mathematical understandings of function composition (e.g., Ayers et al., 1988; Moore & Bowling, 2008). For example, in our prior work we showed how, when given 30 seconds to sketch a graph of the composite of two graphed functions, students did. In this study, we extend our prior work by exploring the ways students reason graphically about the composition of two functions when given time to revise their original sketched graphs, after having sketched graphs to represent the composition of functions represented graphically, algebraically, and tabularly.

Research Question, Background, and Framework

In this study, we seek to answer the following research question: *After having several experiences thinking about function composition, how and why do students revise their responses to graphical function composition tasks?*

Function and Function Composition in Extant Literature

Much literature in mathematics education has been devoted to the function concept (e.g., Breidenbach et al., 1992; Freudenthal, 1983; McCallum, 2019; Mirin et al., 2021; Paoletti et al., 2018). This has included exploring students' conceptions about real-valued functions (from $\mathbb{R}$ to $\mathbb{R}$) through the notion of covariational reasoning (e.g., Paoletti & Moore, 2017). In this context, a deep understanding of function demands students understand how quantities co-vary together – meaning, how changes in one quantity correspond to changes in the other quantity. Studies have shown that students can experience difficulties in understanding the relationship between co-varying quantities and their graphs (e.g., Carlson et al., 2002; Schoenfeld, 1985). More broadly, Dreyfus and Eisenberg (1983) pointed out that without a formula, the graphical representation of function has very little meaning for most students entering calculus.

Yet when it comes to function composition, very little research exists – and what does exist primarily uses algebraic perspectives (e.g., Ayers et al., 1988; Clark et al., 1997; Moore & Bowling, 2008). There is a dearth of literature on function composition, especially through a graphical lens. To address this gap, in a prior analysis (Chen et al., 2023), we identified common types of resultant composite function graphs participants generated in 30 seconds when given graphs, equations, and tables. Although we recognize that students primarily have difficulty with sketching graphs of composite functions given two graphs, in this analysis, we aim to gain further insight into students' reasoning about graphical representations of composite functions. This was done by examining the ways in which students revised their graphs after having further experiences with function composition via equations and tables – which are more familiar ways of thinking about function composition.

Graphical and Algebraic reasoning

A key premise underlying this research is that algebraic and graphical reasonings are different, where each provide distinct conceptions that are complementary for developing deep mathematical meanings. Broadly speaking, algebra is a mathematical field that examines particular structures, based on a set of objects (e.g., $\mathbb{R}$), and a binary operation(s) defined on them (e.g., $+$). Whereas geometry explores spaces that are related with distance, shape, and size (e.g., the Euclidean plane). To highlight this difference, a function, algebraically, is often characterized by its equation, e.g., $f(x) = 2x + 1$ (i.e., the structure of the set of points included in the relation); whereas, graphically, we depict a function with a graph (typically in the Euclidean plane), e.g., $f(x)$ is a line. Moreover, not only do these fields explore different things, but key ways of reasoning differ between them. Driscoll et al. (1999; 2007), for example, differentiated algebraic from geometric "habits of mind." In our context, for simplistic purposes, we use *graphical reasoning* to mean reasoning that relies on figures and shapes in the plane and their properties; *algebraic reasoning*, by contrast, relies on equations.

Methodology and Data Sources

To explore how students reason graphically about function composition, we had university students (n=143) in two mathematics and mathematics education programs (primarily from precalculus classes) sketch the composite function $(g \circ f)(x)$ given two given functions f and g. Participants responded to six tasks (yielding 858 responses), and the task order is important: they went from least familiar to most familiar (in the first four the functions were given as graphs, the next involved two equations, and the last gave a table of values). Students were given the opportunity to revise their original answers after a first attempt at all six tasks – meaning they could sketch a new graph for each of the six tasks after having been refreshed on what were likely more familiar function composition tasks (i.e., the latter using equations and tables).

Task Design

On Desmos, participants were asked to sketch within a limited timeframe (30 seconds, to capture their intuitions and reasonings), the composite function $(g \circ f)(x)$ for six pairs of functions. The majority of these pairs were represented graphically (Figure 1 displays an example Desmos page; Figure 2 displays the other 5 tasks). The purpose in providing participants with somewhat unusual, and non-algebraic forms of the functions, was to deter them from immediately converting to an algebraic equation to complete the composition. Doing so increased the likelihood that students would have to try to reason about function composition graphically based on the shapes of (or relationships represented in) the graphs. In the last two tasks, functions were given in the more familiar forms of an equation and a table. In this paper, we analyze their second attempts – which included either a new composite function graph or a modification of their original graph, as well as a written justification for their changes. Notably, their engagement in these revised tasks happened after participants: i) had more experience with graphing tasks; and ii) had reviewed function composition with – presumably – more familiar modes of reasoning (i.e., equations and tables).

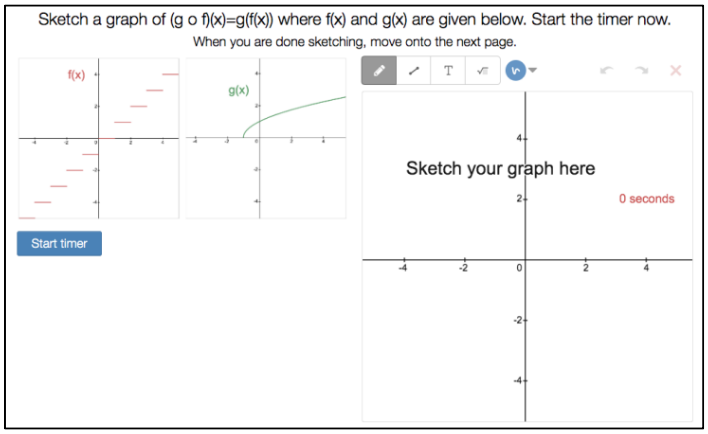


Figure 1. Task 2 in Desmos

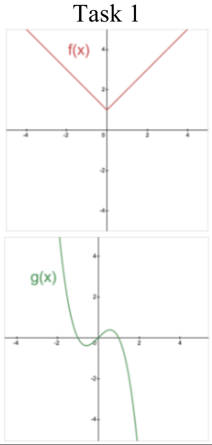
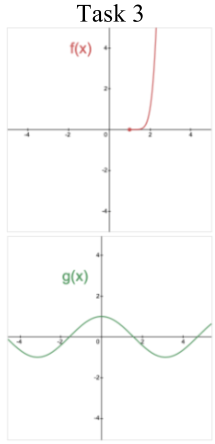
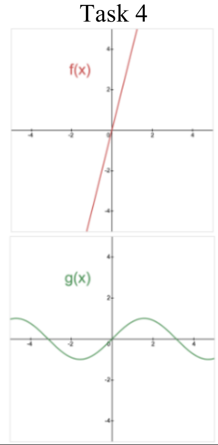
Task 1	Task 3	Task 4	Task 5
			$f(x) = x^2$ $g(x) = \sqrt{x}$
			Task 6 (Presented in Table form) $f(x) = \{(1,4),(2,3),(3,1),(4,-1)\}$ $g(x) = \{(1,-2),(2,0),(3,1),(4,-1)\}$

Figure 2. Given functions for Tasks 1, 3, 4, 5, 6

Analysis and Coding

This study applied grounded theory as the data analysis technique (Stough & Lee, 2021). Out of 85 altered responses (from the possible 858), our analysis focused on (n=73) responses, from 43 distinct participants, in which participants provided (non-blank) justifications. Our analysis identified normative reasoning provided for changes to the initial graphs that were shared across participants. Notably, we aimed to create coding categories that could be applied across the entire set of tasks. We anticipated the revised graphs to include both incorrect and correct graphs of composite functions, and a diverse range of reasoning students may have used. We coded changes between their original and revised graphs as beneficial (+) if they went from incorrect to correct; non-beneficial (-) indicated no progress toward correctness. We generated and refined all categories of coding as a group; when there were uncertainties in how to code a graph, we resolved such cases together and made refinements to categories as needed.

Due to space constraints, we do not include all codes used in our analysis. However, Table 1 provides codes and definitions for the majority of codes – and, in particular, those we report on in this paper. Specifically, these are the codes used to determine students' reasoning for modifying their initial responses. Clarifying examples are given in the results section.

Table 1. Categories of students' reasoning for changing their graph

Code	Description of the reasoning for how or why participants made changes to the composite function $g(f(x))$ sketched
MIXED PROPERTY (MIXED PROP)	One graph put together with the property of the other graph
ALGEBRAIC THINKING (ALG)	Algebraic thinking involving equations or identifying types of functions (e.g., polynomial functions)
DISCRETE POINTS (DP)	Pointwise thinking involving specific points or identifying specific discrete points of the graph (e.g., $f(1)=2$, $g(2)=4$, so $g(f(1))=4$)
TIME	Citing a need for more time on the original tasks

Results

Table 2 provides an overarching summary of the reasoning used by those students who altered their answers after their first attempt. ALGEBRAIC THINKING (ALG) and MIXED PROP were the two dominant responses, so those will be the focus of our results. It is important to note, however, that TIME was also a major factor, with 19 out of 73 responses citing a need for more time on the original tasks as the reason for them altering their original response. Those who ran out of time did not provide any additional reasoning for their answer changes, thus indicating that one reason students revised their graphs was that they needed more time to think about how to compose the two functions. Since a total of 18 out of 73 total changes were beneficial, it reinforces the idea that additional time may provide more opportunities for participants to reason through the challenges and showcase their thinking. There were also a small number of responses (11) that fall under OTHER codes. However, the four most influential categories (ALG, MIXED PROP, DP, and TIME) will be the focus of this report.

Table 2. Summary of data for those with answer changes on Tasks 1-6

Code	Task 1-4	Task 5	Task 6	Total
ALG	4(+)/2(-)	3(+)/6(-)	2(-)	7(+)/10(-)
MIXED PROP	5(+)/14(-)	-	-	5(+)/14(-)
DP	1(+)/2(-)	-	4(+)	5(+)/2(-)
TIME	1(+)/13(-)	2(-)	3(-)	1(+)/18(-)

Note. The symbols (+)/(-) show beneficial/non-beneficial changes in their responses.

Given Graphs

In Tasks 1-4, students were given *graphs* of $f(x)$ and $g(x)$ and asked to sketch the composite function. In these tasks, about 9% of the responses were changed (49 out of 572). See Figures 3 and 4 for (-) and (+) examples (original graph in blue; revised graph in red).

Of those changed responses, 39% of them used MIXED PROP as their reasoning for how they generated their revised graph (19 out of 49). This seems sensible since MIXED PROP implies that students are utilizing and combining physical characteristics of the two graphs – though it was only beneficial in 5 of the 19 cases. On these tasks, 29% (14 out of 49) of responses mentioned timing issues. Perhaps this is because students are accustomed to approaching function composition algebraically, and thus required more time to orient themselves to these graphical representations. Notably, even with the additional time to revise, students still had difficulty – only 1 of the 14 responses that cited time issues made progress toward a correct composite function on these graphical tasks. 12% of students (6 out of 49) still attempted to use ALG to understand these graphical tasks, and this group seemed to make the most progress toward correct graphs. Finally, 6% of students used DP to make sense of the graphical representations (3 out of 49).

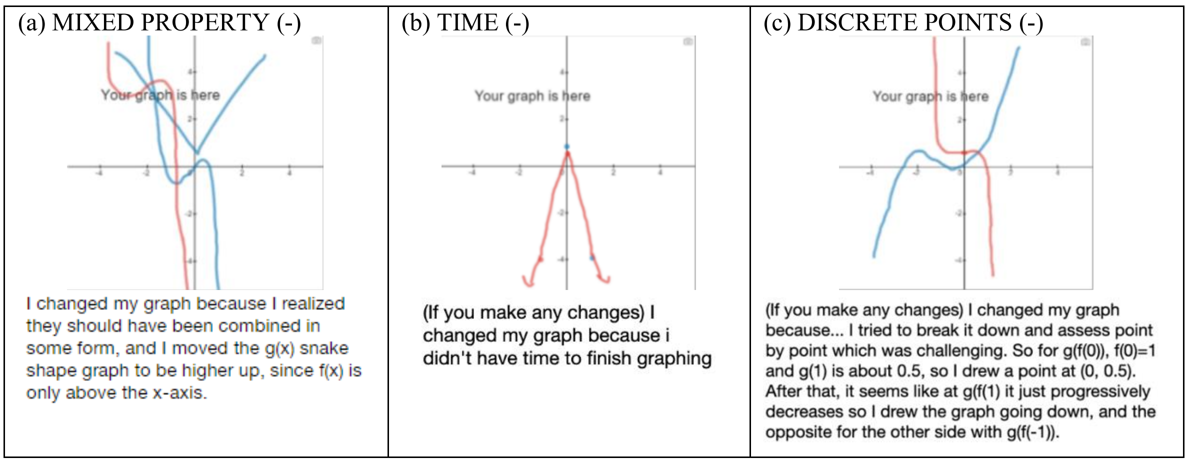


Figure 3. (Still) Incorrect examples from Task 1 of (a) "mixed property" reasoning, (b) "time" reasoning, and (c) "discrete points" reasoning (original graph in blue; revised in red)

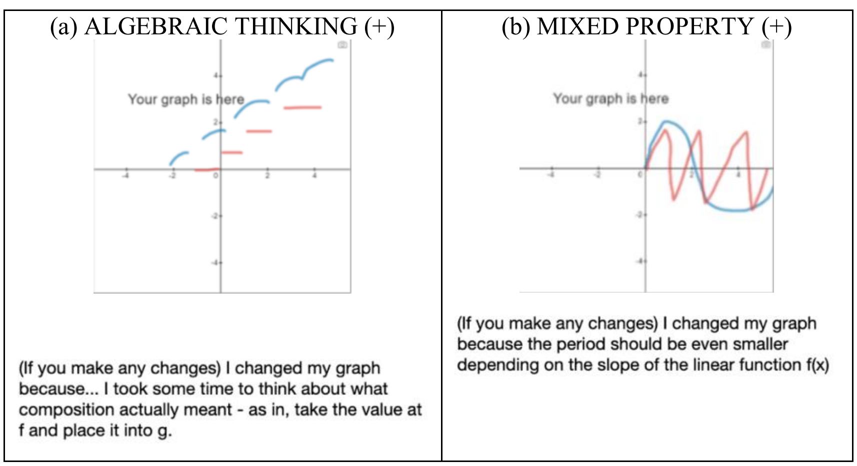


Figure 4. Correct examples from Task 2 of (a) "algebraic thinking" reasoning, and Task 4 of (b) "mixed property" reasoning (original graph in blue; revised in red)

Given Equations

In Task 5, students were given *equations* for $f(x)$ and $g(x)$ and were asked to sketch the composite function. In this task, about 8% of the responses were changed (11 out of 143). See Figure 5 (original in blue; revised in red). Of those changed responses, 82% of them used ALG as their reasoning (9 out of 11), and 18% cited TIME as their reason for changing their response

(2 out of 11). This result is unsurprising, since this task gave students equations, which lends to a more algebraic approach. Of those who changed their answers in Task 5, 27% (3 out of 11) changed their answer so that it improved. All 3 of the improved responses were coded as using ALG for their reasoning, as exemplified in Figure 5a. Interestingly, however, in 4 (of 11) cases students changed their answers from the correct response to the incorrect response. Each was coded as ALG and all were nearly identical to the response in Figure 5c, where x^2 and $\sqrt{x}$ were seen to "cancel" algebraically to x.

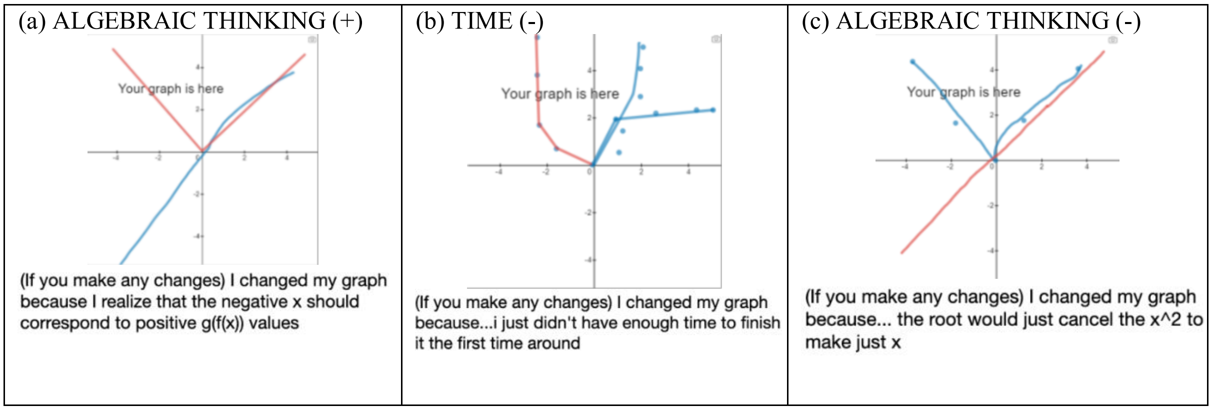


Figure 5. Examples from Task 5 of (a) (c) "algebraic thinking" reasoning, and (b) "time" reasoning (original graph in blue; revised in red)

Given Tables

Task 6 gave students *tables of values* and students were asked to sketch the composite function. In this task, about 9% of the responses were changed when students were given this opportunity (13 out of 143). See Figure 6 for examples (original in blue; revised in red). About 31% of responses (4 out of 13) used DP to explain their changed answers, which makes the most sense based on a table of values. Indeed, all 4 made improvements (see Figure 6a). (Notably, even if students decided to connect the correct set of points with a curve, we considered the composite graph (mostly) correct; the correct graph should just be the discrete set of points.) 23% (3 out of 13) cited an issue with TIME as their reason for changing their answers. 15% (2 out of 13) used ALG to change their answers.

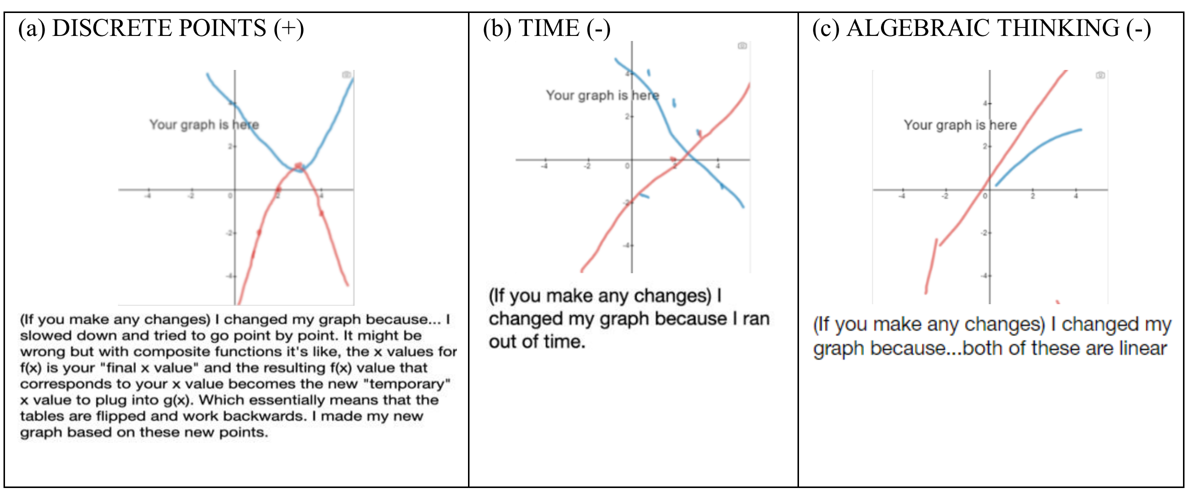


Figure 6. Examples from Task 6 of (a) "discrete points" reasoning, (b) "time" reasoning, and (c) "algebraic thinking" (original graph in blue; revised in red)

Conclusions and Implications

In this section, we highlight four important takeaways for the reader. First, perhaps unsurprisingly, with more time, students performed better. Approximately 25% of the students who edited their responses improved upon their original answer in some way (18 out of 73). This might imply that some students' original wrong responses were due to lack of time rather than lack of ability to perform the composite function task.

Second, when given the opportunity to revise, students seemed to rely primarily on an algebraic approach. Approximately 23% of revised responses cited ALG as their method (17 out of 73). This could imply that more time simply allowed students to rely on the more familiar algebraic manipulations – though, even still, less than half were able to determine the correct composite function. This result seems to suggest that students have not been supported in developing ways of reasoning graphically about function composition, and the extra time serves only to allow for attempts to reason algebraically by identifying approximate equations with which to work.

Third, we also saw that there can be a negative impact to an algebraic approach. All of the students who changed their answers from correct to incorrect (all on Task 5) used algebra in their reasoning for their revisions. This implies that students are using the familiar algebraic approaches even when it actually hinders their natural understanding of the problem! The dual-edged nature of algebraic thinking evident in these revised responses seems to affirm the broad aims of this work – exploring graphical reasoning approaches that support students in developing a richer conception of function composition.

Fourth, overall, the most productive approach in revised responses was based on pointwise thinking (DP) – though most of these were in Task 6. That is, using specific values of x, to evaluate and plot $(x, g(f(x)))$, was helpful reasoning that often led to a correct or mostly correct graphically composed function. Additionally, MIXED PROP seemed to be a promising approach on some "given graphs" (Tasks 1-4) – although identifying which properties of the two functions to combine represents an additional challenge. Both show potential for helping students develop graphical reasoning about function composition.

Collectively, the results presented here along with the results we presented in Chen et al. (2023) highlight that most college students have not been provided opportunities to develop rich meanings for function composition; function composition problems are difficult regardless of if there is a limited time frame and unfamiliar representations (Chen et al., 2023) or if there is an unlimited time frame and students have experienced more familiar representations (this report). Hence, more work is needed to identify further productive intuitions and ways of reasoning students may have that might be helpful to build on. We conjecture novel representations and digital tools might afford learners an opportunity to develop a richer conception of function composition, such as conceiving of function composition graphically or as representing a covariational relationship between multiple quantities. We call for future research to explore this possibility, as well as others, for supporting students in understanding function composition.

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