Towards Operationalizing What Is Learned During Modeling

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In this theoretical paper we leverage constructivist theories of learning to operationalize what is being learned during modeling. We do this by examining model construction through two theories: (i) schemes and concepts and (ii) abstracted quantitative structure. In addition to operationalizing what is being learned, we illustrate our operationalizations by providing examples from STEM undergraduates' model construction activities.

Keywords: mathematical modeling, learning theories, cognition

Scholars have embraced diverse perspectives on mathematical modeling (hereafter, modeling) to address a variety of research problems within the field (Kaiser, 2017). For example, within the cognitive perspective of modeling, researchers have investigated the ways in which to reduce or mitigate the cognitive obstacles students encounter during modeling. Irrespective of the perspective adopted, the researchers share a common goal: improving modelers' learning outcomes in modeling. In order to do this, it is crucial to understand what exactly is being learned during modeling. This knowledge will shed insight onto the instructional strategies and assessment criteria that educators can employ to guide modeling instruction. Two theoretical lenses have been proposed on how modeling enables the learning of mathematical concepts. The models and modeling perspective puts forward the idea that individuals learn through constructing conceptual systems (models) that are used to "construct, describe, or explain the behaviors of other systems that occur in the world' (Lesh et sl., 2003, p. 213). The emergent perspective describes how learning is occasioned as conceptual models are evolved from one context to the other (Gravemeijer, 1999). While these theories are useful to understand how modeling occasions learning, they do not explicate what exactly is being learned during modeling. In this paper, we draw on two distinct (but related) constructivist theories on knowledge construction: (a) schemes and concepts (von Glasersfeld, 1995), and (b) abstracted quantitative structure (Moore et al., 2022) to operationalize what is being learned during modeling. We support our operationalizations with illuminating data from two undergraduate STEM majors' work on modeling tasks.

Theoretical Framework #1: Applying Schemes and Concepts to Modeling

We take the stance that learning occurs through the change of schemes and formation of concepts. von Glasersfeld (1995) identifies three parts of a scheme: (1) An individual's recognition of a certain situation (S); (2) Specific activity associated with that situation (A); (3) and an anticipated result of that activity (R). When a learner sets a goal, it initiates a scheme (S-A-R) (Simon et al., 2004). That is, the goal triggers similar situations (S) where similar goals were made. This recognition is a result of assimilation. The assimilated situation then triggers the specific activity sequence associated with the situation. In effect, this activity leads to a result (say $R*$). If $R = R^*$ (i.e., the result of the activity turns out to be what the learner anticipated), the learner again assimilates. If $R \neq R^*$ (i.e., the result of the activity is not what the learner anticipated), the learner will experience a *perturbation*, also known as a state of disequilibrium. As individuals desire to stay in a state of equilibrium, the learner will attempt to overcome the perturbation through *accommodation*. An individual can accommodate through either (a) modifying the scheme, or (b) re-organizing the scheme. This results in a change in the scheme. (See Figure 1). Therefore, learning has happened, even if the specific change was unintended by the curriculum or teacher.

Von Glasersfeld (1982) describes concept as following:

> “concept” refers to any structure that has been abstracted from the process of experiential construction as recurrently usable, for instance, for the purpose of relating or classifying experiential situations. To be called “concept” these constructs must be stable enough to be re-presented in the absence of perceptual “input” (p. 194).

Hackenberg (2010) makes the case for concept to be “recurrently usable, for instance, for the purpose of relating or classifying experiential situations,” the *abstraction* itself is implemented by the activities of an individual’s schemes. Therefore, Hackenberg (2010) operationalizes concepts as “results of schemes that [individuals] have abstracted from their use of the schemes that the [individuals] can take as given prior to operating… [That is, concepts are] results of schemes that have been *interiorized*” (ibid, p.387). In other words, the results of a scheme, and the activities that produced those results can be taken as given during *assimilatio*n of a new situation. We adopt this view on concepts as well.

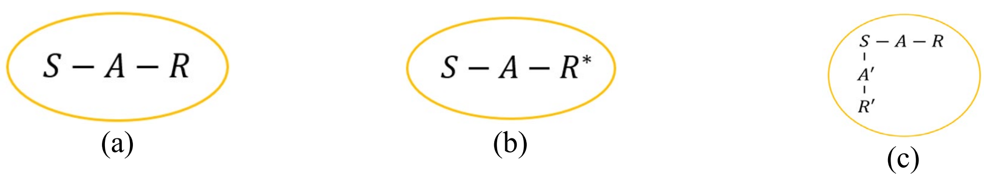


Figure 1. Our Representation for a scheme (a), modification to a scheme (b), and reorganization of a scheme(c)

Object of Validation as Schemes in Use and Standards of Validation as Concepts

Model validation is crucial for mathematical modeling (Czocher, 2018; Zbiek & Connor, 2006) because non-viable models are of little use for solving real-world problems. In Kandasamy & Czocher (2020), we argued that looking deeply into model validation would provide insight into how modeling enables learning because (a) the outcome of model validation may lead to the modification of the model, and (b) learners validate both their final model and their evolving models (p. 924).

When a learner engages in model validation, she holds two models in her mind: the one she is constructing (the object of validation) and the one she anticipates constructing (the standard of validation). The model she is constructing resulted from embracing a goal, which triggered a situation, which triggered a sequence of activities. Thus, the object of validation can be seen as a scheme under use. The model she anticipates is the result of an interiorized scheme and the activities that produced the result. Thus, the standard of validation can be seen as a concept. When the object of validation meets the requirements of the standards of validation, the learner assimilates and remains in a state of equilibrium. When the object of validation does not meet the requirements of the standards of validation, the learner is in a state of disequilibrium. To overcome the perturbation, the learner may choose to either (1) revise the object of validation or (2) accommodate the object of validation through modifying the standards of validation. In these terms, we claim that what modelers learn through modeling are the modifications to the standards of validation. We exemplify this claim using Eshonai’s work on the Cancerous Mass task.

Eshonai and The Cancerous Mass Task

Eshonai was an undergraduate majoring in electrical engineering with a concentration in computer science. In this section, we present a portion of her work on the cancerous mass task, one of 5 modeling problems she solved in a modeling instructional sequence. The interviewer's goal was for Eshonai to build on her knowledge of absolute and relative change to develop a change equation as a model for exponential growth (rate of change is proportional to value of the function). The full task statement included information about the HeLa cell lines and a table of hourly measurements of the sample's mass.

The Cancerous Mass Task (abridged version): Cancer cells can be grown in a lab, for study in their own right and also as a basis for further medical research. The HeLa cell line has a 24-hour propagation rate of 69% of its current mass. Create an expression that would model how quickly the sample is growing.

Using the table of values, Eshonai computed the hourly percent change in mass for the first, second, and third hourly intervals. She obtained 2.93% for each interval. Eshonai concluded that the hourly percent change in mass remained constant as time passes and had a value of 2.93%. She validated her model by computing the hourly percent change for the 22nd time interval. Next, The following conversation between Eshonai and the interviewer was exchanged.

Interviewer: What is the percent change [in mass] during 2-hour segments? So, if you were to look at a segment of time or an interval of time that ranged for 2 hours, lasts for 2 hours, what do you suspect the percent change in mass is?

Eshonai: My gut instinct is saying that it would just be double.

Interviewer: Okay. Double what?

Eshonai: Double the percent change of one hour.

Interviewer: Okay, and why do you suspect this? What is making you suspect that?

Eshonai: Well, the percent change is going to be fixed throughout each individual hour, so I assume that it would be fixed over 2 hours as well.

[Eshonai validated her answer by computing the percent change in mass during $[1, 3]$ and confirmed its "roughly double," as in Figure 2(a)]

Interviewer: What about for a 3-hour increment? What does your gut say?

Eshonai: It would be 3 times. I'll do the math right now.

Eshonai confirmed her hypothesis by evaluating the percent change in mass during $[1,4]$ as shown in Figure 2 (b). After computing the values, she realized that her hypothesis did not hold true and there was a discrepancy between the "actual' value and the "hypothesized" value (Figure 2). Eshonai nominalized the percent change in mass she evaluated through using the percent-change formula as "actual" and the percent change in mass she evaluated through her hypothesis—3 times the percent change in mass during an hour—as "hypothesized."

For 2-hour intervals:

$$\frac{5.29735 - 5}{5} = 5.9463\%$$

$$0.02930224 \cdot 2 = 0.05860$$

(a)

For 3-hour intervals:

$$\frac{5.452539 - 5}{5} = 9.05088$$ *"Actual"*

$$0.02930224 \cdot 3 = 0.087906$$ *"Hypothesized"*

(b)

Figure 2. Eshonai evaluates the percent change in mass during a 2-hour interval (a) and a 3-hour interval.

After computing the "actual" and "hypothesized" values, Eshonai was hesitant to conclude that the percent change in mass during a 3-hour interval was roughly 3 times the percent change in mass during a 1-hour interval. When the interviewer asked her why she thought there was a discrepancy, Eshonai replied, “I’m not sure. I mean, maybe there could be an error in my part. I feel like they should be exactly the same...I have no idea why there is a discrepancy.” With the interviewer’s guidance, Eshonai realized that as the length of the interval increases, her “hypothesized” values further depart from the “actual values”. She modified her claim, asserting that her hypothesis would hold only for small intervals. The interviewer verified, asking if her hypothesis would hold for a half hour interval. Eshonai added, “I would say it is just half [of 2.93%], because if it is straining accuracy with more time [large intervals] then it should be more accurate with smaller of an interval we have.”

In this scenario, Eshonai’s object of validation was *the precent change in mass during the 3-hour interval that she evaluated using the percent change formula*. Eshonai’s standard of validation, the result she was anticipating, was the *percent change in mass during a 3-hour interval is 3 times the percent change in mass during a 1-hour interval*. Her formulation of the standard of validation may have drawn on Eshonai’s prior mathematical experiences with linear growth patterns and was re-confirmed when she computed the percent change in mass during a 2-hour interval (for which she got “roughly” 2 times the percent change in mass during a 1-hour interval). That is, she had interiorized that if the percent changes in mass are the same for each time interval of equal length Δt, then the percent change in mass during N such intervals is N times the percent change in mass during Δt. However, while evaluating the percent change in mass during a 3-hour interval using the percent change formula (the object of validation) she obtained a discrepant value, leading to a perturbation. As a result, she modified her standard of validation by limiting the values of Δt to be small. Therefore, we conjecture, Eshonai’s modification to the standard of validation—her conjecture holds true for only small values of Δt—was what she learned during this particular modeling scenario.

Theoretical Perspective #2: Applying Abstracted Quantitative Structure to Modeling

Moore et al. (2022) reframed von Glasersfeld’s (1982) definition of a concept using quantitative and covariational reasoning theories. Moore et al. (2022) define abstracted quantitative structure as a system of quantitative operations an individual has interiorized. The authors adopt Thomson’s view on quantitative reasoning as conceiving a situation consisting of quantities and relationship among those quantities, where quantities are conceptualized as mental constructs of measurable attributes (Thompson, 2011). Quantitative relationships are a result quantitative operation. The authors operationalize quantitative operations as the mental operations involved in constructing new quantities (Thompson, 1990) and the mental operations involved in reasoning about varying quantities (Carlson et al., 2002). For example, within a disease transmission context, an individual may construct the quantity *how many more people got infected on day 2 than day 1* by additively comparing the quantities: *number of infected people on day 1* and *number of infected people on day 2*. Furthermore, an individual may reason “*as the number of infected people increase, the number of susceptible people decrease*,” engaging in the directional variation of new quantities. Moore et al. (2022) posit that an abstracted quantitative structure has the following characteristics: (1) is recurrently usable beyond its initial experiential construction; (2) can be re-presented in the absence of available perceptual material including that in which it was initially constructed; (3) can be transformed to accommodate to novel contexts permitting the associated quantitative operations; (4) is anticipated as re-presentable in any figurative material that permits the associated quantitative operations (p.44). We argue that modeling activity constructs abstracted quantitative structures, and therefore they are the outcome of learning. We exemplify this claim below.

Pai's Abstracted Quantitative Structure Associated with Interactions

Using Pai's work on an instructional sequence of modeling tasks, we illustrate abstracted quantitative structures as a result of learning during modeling. Pai was an undergraduate senior majoring in economics and minoring in mathematics. The instructional sequence was built to examine participants' quantitative reasoning as they constructed models for real-world situations. The overarching goal of the instructional sequence was to examine how participants' reason quantitatively as they constructed models for real-world situations. Of the 8 tasks Pai worked on, we draw on Pai's work from the following tasks.

The Speed Networking Task (abridged): Suppose you and your friends attend a job fair organized by your department. Each student will have 5-minute "quickfire dates" with a representative from each firm. Create a mathematical model that would represent the total number of meetings between your friends and the representatives from the various firms.

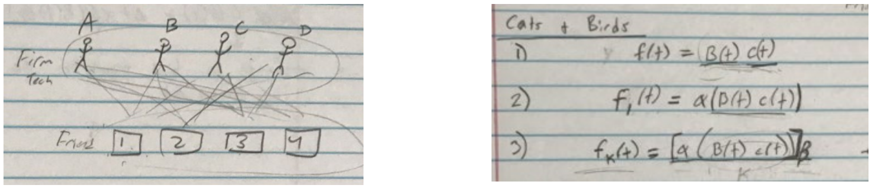
The Cats and Birds Task (abridged): (i) Consider a backyard habitat, where cats are the natural predators of birds. Let $B(t)$ be the number of birds and $C(t)$ be the number of cats at time t. How many potential cat-bird interactions are there at time t? (ii) Not every cat and bird encounter each other. Only some percentage α of potential cat-bird interactions are realized. How would you adapt your model above to account for that fact? (iii) Cats are very good hunters, but they aren't perfect. Sometimes the bird gets away. How would you adapt your model above to represent the number of interactions that results in a bird's death?

The Disease Transmission Task: Suppose a disease is spread by contact between sick and well members of the community. If members of the community move about freely among each other, develop a mathematical model for the dynamics of how the disease would spread through the population.

Below we explain how Pai constructed abstracted quantitative structures associated with interactions between species by providing examples of Pai's initial construction, re-presentation and construction, and finally accommodating to novel contexts.

Initial Construction: The speed networking task. Pai initially modelled this scenario as $y = 5 + 2.5x$, where y represented the total number of meetings between students and representatives and x represented the total number of questions the students may ask from the representative. Pai's model reflects his assumptions about the event's organization: each student would get 5 minutes with each firm, and then an additional 2.5 minutes per question. However, Pai was unsure whether this model satisfied the problem statement. He drew a diagram to represent the meetings between the group of friends and representative from firms (Figure 3a). He counted the number of lines that connected the representatives (A,B,C,D) to the friends (1,2,3,4). Pai then constructed $y = t \cdot f$ to model the number of meetings between friends (f) and representatives (t). Pai used his drawing to justify the model, arguing, "You would simply multiply the number of tech representatives by your number of friends." Pai expressed confidence with the combinatorial model, explicitly qualifying its scope: "every friend talks to every firm, which we cannot know for certain" and "what I have $y = t \cdot f$ is the maximum possible" given that "you cannot have two meetings with the same firm."

In Speed Networking, Pai first constructed an unsatisfactory model. Using his diagram, he was able to construct the quantity *total number of meetings between friends and representatives* as a multiplicative combination of the number of friends and the number of representatives. We argue his activity established a quantitative structure for the total number of possible interactions between two disjoint sets of objects. Next, we argue that he abstracted that structure to carry it to a new scenario.



(a) (b)

Figure 3. Pai's diagram in the Speed Networking task (a) and Pai's models for the Cats & Birds tasks

Re-presenting and constructing: The cats and birds task. After reading the task, Pai immediately wrote the expression $f(t) = B(t)C(t)$ where $f(t)$ represented the total number of potential cat-bird interactions at any time t. Pai stated that the situation was similar to the Speed Networking task. He next modelled the percentage of cat-bird interactions that realized as $f_1(t) = \alpha[B(t)C(t)]$. He justified his expression for the realized cat-bird encounters as "since $f(t)$ is the potential number of cats and birds interactions, assuming they all interact with each other, $f_1(t)$ will be a percentage of that $[B(t)C(t)]$, since not all of them actually interact." He saw a clear link between the subtasks, explaining "I'm assuming each of these models are based on the previous ones."

For the second subtask, Pai first constructed $f_k(t) = \frac{\alpha B(t)C(t)}{K(t)}$ to model the number of cat-bird interactions that resulted in a bird's death. He nominalized $f_k(t)$ as the number of bird-cat interactions that result in a bird's death and $K(t)$ as the number of birds killed at time t. However, he expressed uncertainty around whether $f_1(t)$ should be divided by $K(t)$ or if he should subtract $K(t)$ from $f_1(t)$. The interviewer probed Pai's image of how bird deaths occur, and Pai stated that cats and birds need to meet. He then realized that $f_k(t)$ and $K(t)$ represented the same measurable attribute, number of birds killed due to interaction with cats. Pai then changed his model to $f_k(t) = [\alpha B(t)C(t)]\beta$, where he nominalized β as the "percentage of these interactions [*referring to $\alpha B(t)C(t)$ as he was drawing square brackets, see Figure 3b]* that result in a kill for the cat." Pai justified his revision, arguing that the $B(t)C(t)$ corresponded to total, potential interactions, that $\alpha B(t)C(t)$ corresponded to the actual interactions, and that $[\alpha B(t)C(t)]\beta$ corresponded to the interactions resulting in a dead bird. He concluded, "this makes more sense to me."

Pai wrote $B(t)C(t)$ to model the total number of potential cat-bird interactions without having to draw a diagram, suggesting he re-presented the quantitative structure from Speed Networking in the absence of available perceptual material. The Cats & Birds task presented Pai the opportunity to refine his initial quantitative structure for the total number of interactions between two disjoint sets of objects through removing the assumption that all objects interact exactly once. Pai's model for the number of realized cat-bird interactions, $\alpha B(t)C(t)$, was constructed through shrinking the size of the whole, $B(t)C(t)$, via a scalar α. We interpret that he had constructed a quantitative structure for a subset of the number of interactions. We took Pai's placement of the square brackets in his models (Figure 3(b)) and the excerpt above as evidence that Pai was engaging in shrinking the size of the whole (quantity), in both instances. Next, we show the task that occasioned his accommodation of the quantitative structures.

Accommodating to novel contexts: the disease transmission task. To model the rate of disease spread, Pai introduced the variables $h(t)$ and $I(t)$ to represent the number of healthy people and the number of infected people, respectively. His immediate goal was to model the interaction between healthy and sick people that would result in a healthy person falling sick. He constructed $h(t)I(t)\alpha$, where he defined α to represent "[rate of] successful infections." He justified his model, explaining that "a healthy person interacts with an infected person [gesturing over $h(t)I(t)$], the disease doesn't necessarily spread because they interact. It's going to be some sort of rate, and I have that as α."

In comparison to Speed Networking and Cats & Birds, the Disease Transmission task was a different context, offered less information, and was less scaffolded. Still, Pai set a sub-goal to model the interaction between healthy people and sick people and the interaction that results in a healthy person getting sick. To accomplish this goal, Pai accommodated the quantitative structures he created in the previous task—total number of possible interactions and a subset of the total number of interactions—to this novel context. We take this as indication of Pai having constructed an abstracted quantitative structure associated with the number of interactions between two disjoint set of objects and subsets of those interactions, as it satisfies the characteristics proposed by Moore at al. (2022). Consequently, we claim that through engaging with the instructional sequence of modeling tasks, Pai learned the number of interactions between disjoint set of objects can be constructed via multiplicatively combining the number of each objects and that a subset of interactions can be constructed by scaling the size of the whole.

Discussion

In this paper, we operationalized what is being learned during modeling through (i) schemes and concepts and (ii) abstracted quantitative structures. Through the first lens, we argued that modifications to the standards of validation are what is being learned during modeling. Through the second lens we made the case that learners' construction of abstracted quantitative structures is what is learned during modeling. Our analysis sheds insights into concretizing the end goal of modeling, in terms of learning. This has implications: first, knowing what is learned during modeling will help educators to design modeling tasks that engender that learning. For example, when adopting the view that modifications to the standards are learned during modeling, tasks can be designed to challenge learners' existing standards of validation, providing learners the opportunity for accommodation. Second, the knowledge of what is being learned during modeling is instrumental for educators to be intentional with their instructional goals for modeling and learning environments that support those goals, mitigating some of the inadvertent cognitive obstacles learners may experience during modeling. Finally, it gives educators a sense of what to expect when assessing learners' modeling competencies. We close with two considerations: limitations and future directions. Using the schemes and concepts theoretical orientation, we neither analyzed what reorganizing a standard of validation may look like nor what is learned in the absence of perturbation. Additionally, as standards of validation reside in the mind of the individual, it is not always possible to observe or make predictions about learners' standards of validations nor delineate between the object and standard of validation. To attend to this, future interview protocols can include specific questions regarding learners' model validation such as, *what made you revise your model? Were you anticipating some other result? What was your goal in revising?* However, this theoretical paper is a significant contribution to the field as it initiates a discourse on leveraging established theories of learning to effectively operationalize the learning goal of modeling, and the instructional strategies for nurturing the learning of modeling both as a means and as an end.

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