Raise Your Hand if You Are Overloaded: The Role of Gestures in Proofs Comprehension

Vladislav Kokushkin
Colorado State University

Activities related to reading and understanding mathematical proofs are notoriously challenging for college students. On top of applying advanced logico-mathematical knowledge, these activities require the reader to process and operate on a substantial amount of information. Failure to successfully navigate the incoming stream of information may result in cognitive overload and impede one's ability to make progress on the task. This case study investigates how an undergraduate student, David, used his hands to overcome the cognitive load he experienced when reading a proof of the Two-Color Theorem. The findings suggest that in the absence of other modes of offloading (e.g., using pen and paper, figurative materials, or technology), hand gesturing may serve as a powerful and convenient offloading mechanism.

Keywords: gestures, working memory, proofs

The ability to read and comprehend proofs is critical for undergraduate students' success in advanced mathematics courses. Students are expected to spend substantial time surveying proofs presented in mathematics textbooks and their instructor's lecture notes (Weber, 2004). However, the activity of reading mathematical proofs differs from reading a literary text and requires developing special experience and particular skills. Unsurprisingly, then, numerous studies have documented that college students struggle to read proofs effectively (e.g., Hodds et al., 2014; Inglis & Alcock, 2012; Mejía-Ramos et al., 2012; Selden & Selden, 2003; Weber & Alcock, 2005; Weber 2004, 2015).

The implementation of certain ineffective reading strategies may be associated with the need to operate on an overwhelming amount of information. For example, Selden and Selden (2003) observed that novice readers spend much time paying attention to surface features of proofs, such as algebraic calculations. In another study, Inglis and Alcock (2012) found that undergraduate students tend to read proofs line-by-line, or, in other words, implement a *zooming in* (Weber & Mejía-Ramos, 2011) strategy. This way of reading proofs is the opposite of a *zooming out* strategy – skimming the proof to grasp the main ideas prior to attending to the details. The tendency to read proofs line-by-line or focus on surface details may pose significant cognitive load (Sweller, 1988, 2010) and impede proofs comprehension.

Cognitive offloading refers to the use of embodied activities to reduce the experienced cognitive load (Risko & Gilbert, 2016). Mathematics education literature has documented the beneficial effects of different modes of cognitive offloading, for example, drawings (Stryker et al., 2022), and hand gesturing (e.g., Cook et al., 2013; Goldin-Meadow et al., 2001). Remarkably, all these studies were conducted in the context of elementary school mathematics, and the role of cognitive offloading as manifested via hand gesturing in the context of undergraduate mathematics has been fundamentally under-researched. Moreover, none of the previous studies investigated the role of gestures in understanding written mathematical texts. As such, the goal of this study is to begin filling the gaps in the RUME literature. Specifically, my chief research question is *What is the role of students' gestures in navigating the cognitive load associated with reading and comprehending mathematical proofs?*

Theoretical Framework

In answering this research question, I adopt Piagetian and embodied perspectives. Embodied theories of cognition (e.g., Barsalou, 2008; Lakoff & Nuñez, 2000; Nemirovsky & Ferrara, 2009) propose that cognitive activities are rooted in sensory-motor processes (Wilson, 2002). Likewise, the idea that sensory-motor operations provide a basis for mathematical learning corresponds to Piagetian theories of mathematical development (Piaget, 1972). Sensory-motor activity is a kinesthetic activity involving muscular movements, including hand gestures. This activity can be further internalized and transformed into mental actions and representations. Although Piagetian and embodied perspectives have certain epistemological disagreements (e.g., on how abstraction takes place), these two theories allow for the integration of the constructs of mathematical proofs, gestures, and working memory (WM) by positioning bodily activities at the core of human cognition.

Working Memory

WM is a psychological model for human's ability to simultaneously store and process limited amount of information over a short period of time (e.g., Baddeley, 1992; Baddeley & Hitch, 1974; Daneman & Carpenter, 1980). Although numerous models of WM have been proposed (e.g., Atkinson & Shiffrin, 1968; Baddeley & Hitch, 1974; Cowan, 2012), Pascual-Leone's (1970) neo-Piagetian approach to WM is of primary relevance to my theoretical framework.

Pascual-Leone (1970, 1987) introduced a mathematical model to account for the development and functioning of WM. The proposed model has two levels. The first level contains cognitive units – schemes in the Piagetian sense. The second level includes cognitive operators, which are responsible for the activation of task-relevant schemes and inhibition of activation of irrelevant schemes. An individual's WM capacity (or *M-capacity*) is characterized in terms of the maximal number of schemes they can activate at a time. The cognitive complexity of the task (its *M-demand*) is defined as the number of schemes the task requires to be activated simultaneously. If the M-demand of the task exceeds an individual's M-capacity, cognitive overload takes place, and student's progress on the task can be impeded.

Gestures

Hostetter and Alibali (2008, 2019) proposed the Gesture as Simulated Action (GSA) framework to provide a theoretical account of how gestures emerge from embodied simulations of motor and perceptual states. In line with neurological and behavioral evidence, the GSA framework asserts that thinking and talking about an action activates neural areas that are responsible for planning the action. This activation may be either inhibited by a human or realized as a sensory-motor output. In the latter case, the hand gesture is born.

Mathematical Proofs

Harel and Sowder (2005, 2007) have proposed the notion of *transformational proof schemes.* According to this view, when students read proofs, their reasoning is transformational in nature, because learners are supposed to perform various transformations of mathematical objects of thought to deduce or keep track of a chain of inferences. Although the authors originally defined transformations only as expressed through written and spoken language, I follow Pier et al. (2019) and extend this definition by including transformations manifested via hand gesturing.

Methods

The data presented here are a subset of data collected for a larger project aimed at understanding the role of gestures in offloading cognitive demands on students' WM when they read, present, and construct proofs of mathematical conjectures. In this paper, I report on David, an undergraduate student enrolled in an introductory proof course, working on the proof of the Two-Color Theorem (see Figure 1).

Theorem: *$n \geq 1$ circles are given in the plane. They divide the plane into regions. Then it is possible to color the plane using two colors, so that no two regions with a common boundary line are assigned the same color.*

Proof: call the proposition $P(n)$. Let the two colors be B and W. We will prove the theorem using proof by mathematical induction.

Base case: for $n = 1$, the result $P(1)$ is clear. If there is only one circle, we may color the inside B and the outside W, and this coloring satisfies the requirements of the theorem.

Inductive hypothesis: assume that $P(k)$ holds for a fixed but arbitrary k. So we know that for any k circles there is a coloring which satisfies the requirements of the theorem.

Inductive step: we need to show that the proposition holds for $k+1$. Consider $k+1$ circles $\{C_1, C_2, \ldots, C_{k+1}\}$. Ignoring the circle C_{k+1}, we temporarily have k circles $\{C_1, C_2, \ldots, C_k\}$ By the inductive hypothesis, the regions created by $C_1, ..., C_k$ can be colored according to the requirements of the theorem. We color the plane according to this coloring. Now we add the circle C_{k+1} back and do the following

1. For any region which lies inside C_{k+1}, do not change its color;
2. For any region which lies outside C_{k+1}, recolor it with the opposite color.

Now we may check that the new coloring works:

1. Two neighboring regions whose boundary lies inside C_{k+1} have different colors;
2. Two neighboring regions whose boundary lies outside C_{k+1} have different colors;
3. Two neighboring regions whose boundary lies on C_{k+1} have different colors.

This shows that $P(k+1)$ is true, and so by the principle of induction, the proof follows. ■

Figure 1. The proof of the Two-Color Theorem (as it was presented to the participant).

Procedures

David participated in two pre-interview WM assessments, a proof-based clinical interview, and a stimulated recall interview (SRI).

WM assessments. The participant's WM capacity was measured using two previously validated procedures – the Figural Intersection Test (Pascual-Leone & Ijaz, 1989) and the Direct Following Task (Pascual-Leone & Johnson, 2005). David's scores on the two tests were averaged and resulted in a WM capacity of 8.25[1].

Proof-based interview. The participant was presented with a proof printed out on the piece of paper as it is shown in Figure 1. The student was told that the proof was correct and that his

[1] Average WM capacity ranges between 5 and 9 (Miller, 1956; Pascual-Leone, 1970).

task was to read the proof and do his best to understand it. The student was not given a pen, scratch paper, a calculator, or any other figurative materials. He was not limited in time and was asked to think aloud while surveying the proof. After David indicated that he fully understood the proof, the interviewer asked a set of comprehension questions (data not reported here). The participant's behavior and mathematical reasoning were video- and audio-recorded.

Stimulated recall interview. On a later date, the student was shown selected video clips with himself working on the proof, and the interviewer asked questions about the participant's reasoning and behavior.

Hypothetical Analysis of the M-demand of the Proof

Prior to collecting data, I conducted a hypothetical analysis of the M-demand of the proof. This analysis is based on identifying a sequence of mental actions that need to be carried out to understand the proof. Specifically, I broke the proof into lines containing one or two mathematical statements. Every line was then coded as A_k. For each line, I identified the set of 1) previous lines A_k logically depends on, and 2) contextual schemes that a participant would hypothetically need to activate when reading the proof. These contextual schemes included extra lemmas, definitions, key concepts (e.g., primality or divisibility), and the essential elements of the proof framework (e.g., inductive hypothesis or proof by cases). For the purposes of this project, I use the word "scheme" in a broad sense to capture the information contained in the text of the proof (lines $A_1, \ldots, A_n$) and contextual mathematical knowledge.

Figure 2 illustrates the hypothetical M-demand of the proof of the Two-Color Theorem. It has been hypothesized that to understand the line A_7, the reader needs to activate the schemes A_3, A_4, and A_5. Also, the reader needs to allocate additional cognitive resources to retain the information contained in A_7 itself and to understand why $\{A_3, A_4, A_5\}$ imply A_7 (indicated as an arrow). Thus, the M-demand of line A_7 was assessed as 5 (one unit per scheme plus an arrow). According to the same methodology the M-demand of the lines $A_9 – A_{11}$ was assessed as 9, which is higher than an average WM capacity.

A_3: By the inductive hypothesis, the regions created by $C_1, ..., C_k$ can be colored according to the requirements of the theorem. $[\{A_2, P, IH\} \to A_3]$

A_4: We color the plane according to this coloring. $[A_3 \to A_4]$

A_5: Now we add the circle C_{k+1} back and do the following $[\{A_1, A_2, A_4\} \to A_5]$

A_6: 1. For any region which lies inside C_{k+1}, do not change its color; $[\{A_3, A_4, A_5\} \to A_6]$

A_7: 2. For any region which lies outside C_{k+1}, recolor it with the opposite color. $[\{A_3, A_4, A_5\} \to A_7]$

A_8: Now we may check that the new coloring works: $[\{A_3, A_4, A_5, A_6, A_7\} \to A_8]$

A_9: 1. Two neighboring regions whose boundary lies inside C_{k+1} have different colors; $[\{A_3, A_4, A_5, A_6, A_7, A_8, Cs\} \to A_9]$

A_{10}: 2. Two neighboring regions whose boundary lies outside C_{k+1} have different colors; $[\{A_3, A_4, A_5, A_6, A_7, A_8, Cs\} \to A_{10}]$

A_{11}: 3. Two neighboring regions whose boundary lies on C_{k+1} have different colors. $[\{A_3, A_4, A_5, A_6, A_7, A_8, Cs\} \to A_{11}]$

Figure 2. A line-by-line analysis of the M-demand of the proof of the Two-Color Theorem. The scheme P denotes the proposition of the theorem, IH – the inductive hypothesis, and Cs – the necessity to coordinate cases.

Data Analysis

The analysis began with creating the transcript of the proof-based interview. The videos in conjunction with the written transcript were used to identify gesturing episodes. Given that gestures oftentimes occurred quickly, I re-watched the video recordings and created codes for all gestures. I did this by looking at each gesture and creating a short description considering physical hand movements, the associated mathematical context, and the corresponding speech (if present). Gestures that did not seem to convey any mathematical meaning, were eliminated from the analysis. The remaining gestures were then labeled as (purely) communicative or offloading. In the latter case, the offloaded schemes were specified. The participant's responses during the SRI were used to triangulate my initial interpretations of his gesturing.

Results

I chose the Two-Color Theorem for two main reasons. First, its proof is mathematically accessible for undergraduate students assuming that they are familiar with the method of mathematical induction. Second, my methodology for measuring the M-demand of the proof predicted significant cognitive overload even for the participants with advanced WM capacity.

As was expected, David did not experience difficulties with the task until he reached the end of the proof. However, when reading lines $A_9 – A_{11}$, the participant demonstrated behavioral indicators of cognitive overload. Specifically, he paused reading and reviewed the details of (re)coloring of the plane (lines $A_4 – A_7$) multiple times, asked for a pen, and eventually started assisting himself with active hand gesturing to represent the circles and the associated regions:

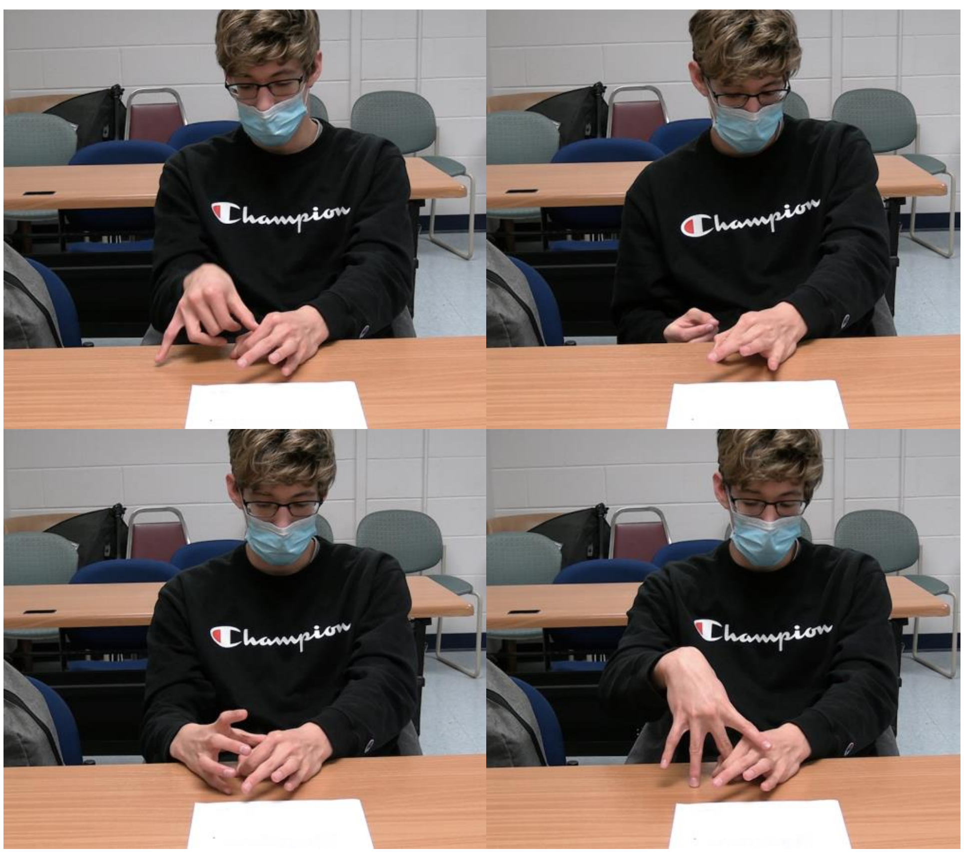

Figure 3. David working on the proof of the Two-Color Theorem.

> So, this is a circle [uses his thumb and a pointing finger to form a circle]. And it has a different color from its outer region [points to the inside and outside of the circle, Figure 3, top left]. C_k [meaning C_{k+1}] is put on top of it [uses another hand to represent a circle, Figure 3, top right]. And now it has an opposite color from the original circle [Figure 3, bottom left]. And it is a neighboring region, with the original outer region, which was opposite of its inside [points to the inside of the original circle and to the outer region. Figure 3, bottom left, then bottom right]. They cannot be both opposite to an inside circle… Oh wait, but they switch the color!

The described sequence of gestures allowed David to offload the activation of a few critical schemes and helped him to make progress in understanding the proof. By placing the first circle and pointing to the inner/outer regions, the student offloaded schemes A_3 and A_4. When using his hand to indicate the action of adding another circle on top of the first one, he transmitted the activation of scheme A_5 into his immediate sensory-motor experience and eventually realized that some recoloring was needed, which is one of the key ideas of the proof. Therefore, offloading schemes A_3, A_4, and A_5 allowed the participant to free up WM resources to operate on other pieces of information, such as scheme A_7.

In the SRI, I showed David the described video episode and asked him what was challenging about the proof and what helped him to overcome these challenges. The participant indicated that it was hard to "remember all the configurations and steps" and discussed some offloading effects of hand gesturing:

> …maybe honestly, the working memory tasks helped me realize that if I could like potentially store some of the memory, like, in my hands, and not move them, and now it helped me like, work on other stuff in my head[2]. So, I just froze that piece of the memory in my hands and then continued to like, think about other configurations… Because, um... it was easier to like visualize different colorings of some regions I can actually see. Because I didn't have to remember both the colors and the region, I just had to remember that.

Although David's reflection could be biased by his prior knowledge about WM, which relates to the context of the study, these data provide an important insight into the nature of the offloading power of hand gestures from the participant's perspective.

Discussion and Implications

My work contributes to the existing literature in multiple ways. First, this case study serves as a starting point for the investigation of the role of students' gestures in navigating the cognitive load associated with activities pertaining to high-level mathematical cognition, such as mathematical proofs. In line with the previous research conducted in the context of elementary mathematics (e.g., Alibali & DiRusso, 1999; Cook et al., 2012; Goldin-Meadow et al., 2001; Ping & Goldin-Meadow, 2010; Wagner et al., 2004), the data presented here provide evidence that, in the absence of other modes of offloading, gesturing may be a convenient and powerful offloading mechanism. Using his hands to offload the activation of task-relevant mental

---

[2] After the WM assessments, David was curious about what constitutes WM and why it is important in learning mathematics. Apparently, he picked up some WM terminology and used it in the SRI.

schemes, David could reduce the experienced cognitive load and free up WM resources that helped him to make progress in proof comprehension.

Second, prior research on gesturing in proofs has revealed the prevalence of so-called dynamic gestures during proving practices (e.g., Nathan & Walkington, 2017; Nathan et al., 2021; Walkington et al., 2014). These gestures depict real-time transformations of either a single object or multiple mathematical entities, related to each. Dynamic gestures were also evident in my data (for example, gestures depicting the transformations of the regions caused by adding the k+1st circle onto the plane). The findings shed light on the cognitive utility of dynamic gestures in terms of their offloading power.

Finally, the proposed analysis of hypothetical M-demands of mathematical proofs was able to model the experienced cognitive load and predict places of cognitive overload by specifying the number of mental schemes the proof requires to be activated line-by-line. Other scholars used alternative approaches for modeling the cognitive load of a mathematical task. For example, Norton et al. (2023) introduced the notion of unit transformation graphs (UTGs) to illustrate the sequence of mental actions students perform when they are engaged in fractional tasks. Although UTGs provide opportunities for measuring the experienced cognitive load and help to explain how it may be reduced (Norton et al., 2023; Stryker et al., 2022), the applicability of this model is limited, and UTGs become impractical in multidimensional tasks, such as mathematical proofs.

As a closing remark, hand gesturing is only one, although outstanding, example of a cognitive offloading mechanism. In another study, Stryker and colleagues (2022) examined the role of student-generated drawings in offloading cognitive demands of fractional tasks. The results indicated that the possibility to rely on drawings helped the students to free up WM resources and complete a cognitively demanding task. Notably, before starting drawing, the participants actively produced hand gestures. However, the use of gesturing did not help the students to overcome the experienced cognitive overload and finish the task without drawing. This gives some anecdotal evidence that drawing can be considered a "more powerful" offloading tool than gesturing. Nevertheless, future research should directly contrast various modes of offloading.

References

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive Development*, 14, 37–56.

Atkinson, R. C., & Shiffrin, R. M. (1968). Human memory: A proposed system and its control processes. In K. W. Spence & J. T. Spence (Eds.), The psychology of learning and motivation: Advances in research and theory (Vol. 2, pp. 89–195). New York, NY: Academic Press.

Baddeley, A. D., & Hitch, G. (1974). Working memory. In *Psychology of learning and motivation* (Vol. 8, pp. 47–89). Academic press.

Baddeley, A. (1992). Working memory. *Science*, *255*(5044), 556–559.

Barsalou, L. W. (2008) Grounded cognition. *Annual Review of Psychology, 59*, 617– 645.

Cook, S. W., Yip, T. K., & Goldin-Meadow, S. (2012). Gestures, but not meaningless movements, lighten working memory load when explaining math. *Language and cognitive processes*, *27*(4), 594–610.

Cowan, N. (2012). *Working memory capacity*. Psychology press.

Daneman, M., & Carpenter, P. A. (1980). Individual differences in working memory and reading. *Journal of verbal learning and verbal behavior*, *19*(4), 450–466.

Goldin-Meadow, S., Nusbaum, H., Kelly, S., & Wagner, S. (2001). Explaining math: Gesturing lightens the load. *Psychological Science*, 12, 516–522.

Harel, G., & Sowder, L. (2005). Advanced mathematical thinking at any age: Its nature and its development. *Mathematical thinking and learning*, *7*(1), 27–50.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. *Second handbook of research on mathematics teaching and learning*, *2*, 805–842.

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, *45*(1), 62–101.

Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic bulletin & review*, *15*(3), 495–514.

Hostetter, A. B., & Alibali, M. W. (2019). Gesture as simulated action: Revisiting the framework. *Psychonomic bulletin & review*, *26*(3), 721–752.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*(4), 358–390.

Lakoff, G., & Núñez, R. E. (2000). *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. New York, NY, USA: Basic Books.

Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, *79*(1), 3–18.

Miller, G. A. (1956). The magical number seven, plus or minus two: some limits on our capacity for processing information. *Psychological Review, 63*(2), 81–97.

Nathan, M. J., Schenck, K. E., Vinsonhaler, R., Michaelis, J. E., Swart, M. I., & Walkington, C. (2021). Embodied geometric reasoning: Dynamic gestures during intuition, insight, and proof. *Journal of Educational Psychology, 113*(5), 929–948.

Nathan, M. J., & Walkington, C. (2017). Grounded and embodied mathematical cognition: Promoting mathematical insight and proof using action and language. *Cognitive research: principles and implications*, *2*(1), 1–20.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, *70*(2), 159–174.

Norton, A., Ulrich, C., & Kerrigan, S. (2023). Unit transformation graphs: modeling students' mathematics in meeting the cognitive demands of fractions multiplication tasks. *Journal for Research in Mathematics Education, 54*(4), 240–259.

Pascual-Leone, J. (1970). A mathematical model for the transition rule in Piaget's developmental stages. *Acta psychologica*, *32*, 301–345.

Pascual-Leone, J. (1987). Organismic processes for neo-Piagetian theories: A dialectical causal account of cognitive development. *International Journal of Psychology*, *22*(5-6), 531–570.

Pascual-Leone, J., & Ijaz, H. (1989). Mental capacity testing as a form of intellectual-developmental assessment. *Assessment and placement of minority students*, 144–171.

Pascual-Leone, J., & Johnson, J. (2005). Handbook of Understanding and Measuring Intelligence. India: SAGE Publications.

Piaget, J. (1972). Intellectual evolution from adolescence to adulthood. *Human development, 15*(1), 1–12.

Pier, E. L., Walkington, C., Clinton, V., Boncoddo, R., Williams-Pierce, C., Alibali, M. W., & Nathan, M. J. (2019). Embodied truths: How dynamic gestures and speech contribute to mathematical proof practices. *Contemporary Educational Psychology*, *58*, 44–57.

Ping R.M., Goldin-Meadow S. (2010). Gesturing saves cognitive resources when talking about nonpresent objects. *Cognitive Science,* 34, 602–619.

Risko, E. F., & Gilbert, S. J. (2016). Cognitive offloading. *Trends in cognitive sciences*, *20*(9), 676–688.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for research in mathematics education*, *34*(1), 4–36.

Stryker, R., Kokushkin, V., Kerrigan, S., & Norton, A. (2022). Offloading cognitive demands of fractional tasks on working memory through drawings (pp. 1626-1635). *Proceedings of the Forty-fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Nashville, TN.

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science, 12,* 257–285.

Sweller, J. (2010). Cognitive load theory: Recent theoretical advances. In J. L. Plass, R. Moreno, & R. Brünken (Eds.), *Cognitive load theory* (pp. 29–47). Cambridge University Press.

Wagner, S. M., Nusbaum, H., & Goldin-Meadow, S. (2004). Probing the mental representation of gesture: Is handwaving spatial? *Journal of Memory and Language*, *50*(4), 395–407.

Walkington, C., Boncoddo, R., Williams, C., Nathan, M. J., Alibali, M. W., Simon, E., & Pier, E. (2014). Being mathematical relations: Dynamic gestures support mathematical reasoning. Boulder, CO: International Society of the Learning Sciences.

Weber, K. (2004) Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course, *The Journal of Mathematical Behavior, 23*(2), 115–133.

Weber, K. (2015). Effective proof reading strategies for comprehending mathematical proofs. *International Journal of Research in Undergraduate Mathematics Education*, *1*(3), 289–314.

Weber, K., & Alcock, L. (2005). Using warranted implications to understand and validate proofs. *For the Learning of Mathematics*, *25*(1), 34–51.

Weber, K., & Mejia-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics*, *76*(3), 329–344.

Wilson, M. (2002). Six Views of Embodied Cognition. *Psychonomic Bulletin & Review*, *9*(4), 625–636.